

# Quasi-Valuations Extending a Valuation

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**ABSTRACT.** Suppose  $F$  is a field with valuation  $v$  and valuation ring  $O_v$ ,  $E$  is a finite field extension and  $w$  is a quasi-valuation on  $E$  extending  $v$ . We study quasi-valuations on  $E$  that extend  $v$ ; in particular, their corresponding rings and their prime spectrums. We prove that these ring extensions satisfy INC (incomparability), LO (lying over), and GD (going down) over  $O_v$ ; in particular, they have the same Krull Dimension. We also prove that every such quasi-valuation is dominated by some valuation extending  $v$ .

Under the assumption that the value monoid of the quasi-valuation is a group we prove that these ring extensions satisfy GU (going up) over  $O_v$ , and a bound on the size of the prime spectrum is given. In addition, a 1:1 correspondence is obtained between exponential quasi-valuations and integrally closed quasi-valuation rings.

Given  $R$ , an algebra over  $O_v$ , we construct a quasi-valuation on  $R$ ; we also construct a quasi-valuation on  $R \otimes_{O_v} F$  which helps us prove our main Theorem. The main Theorem states that if  $R \subseteq E$  satisfies  $R \cap F = O_v$  and  $E$  is the field of fractions of  $R$ , then  $R$  and  $v$  induce a quasi-valuation  $w$  on  $E$  such that  $R = O_w$  and  $w$  extends  $v$ ; thus  $R$  satisfies the properties of a quasi-valuation ring.

## §0 INTRODUCTION

Recall that a valuation on a field  $F$  is a function  $v : F \rightarrow \Gamma \cup \{\infty\}$ , where  $\Gamma$  is a totally ordered abelian group and  $v$  satisfies the following conditions:

- (A1)  $v(0) = \infty$ ;
- (A1')  $v(x) \neq \infty$  for every  $0 \neq x \in F$ ;
- (A2)  $v(xy) = v(x) + v(y)$  for all  $x, y \in F$ ;
- (A3)  $v(x + y) \geq \min\{v(x), v(y)\}$  for all  $x, y \in F$ .

Valuation theory has long been a key tool in commutative algebra, with applications in number theory and algebraic geometry. It has become a useful tool in the study of division algebras, and used in the construction of various counterexamples such as Amitsur's construction of noncrossed products division algebras. See [Wad] for a comprehensive survey.

Generalizations of the notion of valuation have been made throughout the last few decades. Knebusch and Zhang (cf. [KZ]) have studied valuations in the sense of Bourbaki [Bo, section 3]. Thus they were able, omitting (A1'), to study valuations on any commutative ring rather than just on an integral domain. They define the notion of Manis valuation (the valuation is onto the value group) and show that an  $R$ -Prüfer ring is related to Manis valuations in much the same way that a Prüfer domain is related to valuations of its quotient field.

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\*This paper is part of the author's forthcoming doctoral dissertation.

A monoid  $M$  is called a *totally ordered* monoid if it has a total ordering  $\leq$ , for which  $a \leq b$  implies  $a + c \leq b + c$  for every  $c \in M$ . When we write  $b > a$  we mean  $b \geq a$  and  $b \neq a$ .

In this paper we study quasi-valuations, which are generalizations of valuations. A *quasi-valuation* on a ring  $R$  is a function  $w : R \rightarrow M \cup \{\infty\}$ , where  $M$  is a totally ordered abelian monoid, to which we adjoin an element  $\infty$  greater than all elements of  $M$ , and  $w$  satisfies the following properties:

- (B1)  $w(0) = \infty$ ;
- (B2)  $w(xy) \geq w(x) + w(y)$  for all  $x, y \in R$ ;
- (B3)  $w(x + y) \geq \min\{w(x), w(y)\}$  for all  $x, y \in R$ .

In the literature this is called a pseudo-valuation when the target  $M$  is a totally ordered abelian group, usually taken to be  $(\mathbb{R}, +)$ . As examples one can mention Cohn (cf. [Co]) who gave necessary and sufficient conditions for a non-discrete topological field to have its topology induced by a pseudo-valuation; Huckaba (cf. [Hu]) has given necessary and sufficient conditions for a pseudo-valuation to be extended to an overring, and Mahdavi-Hezavehi has obtained "matrix valuations" from matrix pseudo-valuations (cf. [MH]). We use the terminology quasi-valuation to stress that the target monoid need not be a group. Moreover, our study concentrates on quasi-valuations extending a given valuation on a field.

The minimum of a finite number of valuations with the same value group is a quasi-valuation. For example, the  $n$ -adic quasi-valuation on  $\mathbb{Q}$  (for any positive  $n \in \mathbb{Z}$ ) already has been studied in [Ste]. (Stein calls it the  $n$ -adic valuation.) It is defined as follows: for any  $0 \neq \frac{c}{d} \in \mathbb{Q}$  there exists a unique  $e \in \mathbb{Z}$  and integers  $a, b \in \mathbb{Z}$ , with  $b$  positive, such that  $\frac{c}{d} = n^e \frac{a}{b}$  with  $n \nmid a$ ,  $(n, b) = 1$  and  $(a, b) = 1$ . Define  $w_n(\frac{c}{d}) = e$  and  $w_n(0) = \infty$ .

A quasi-valuation is a much more flexible tool than a valuation; for example, quasi-valuations exist on rings on which valuations cannot exist.

Three main classes of rings were suggested throughout the years as the non-commutative version of a valuation ring. These three types are invariant valuation rings, total valuation rings, and Dubrovin valuation rings. They are interconnected by the following diagram:

$$\{\text{invariant valuation rings}\} \subset \{\text{total valuation rings}\} \subset \{\text{Dubrovin valuation rings}\}.$$

Morandi (cf. [Mor]) has studied Dubrovin valuation rings and their ideals. It turned out that unlike a valuation on a field, the value group of a Dubrovin valuation ring  $B$  does not classify the ideals in general but does so when  $B$  is integral over its center.

At the outset and in section 1 one sees that there are an enormous amount of quasi-valuations, even on  $\mathbb{Z}$ , so in order to obtain a workable theory one needs further assumptions. Morandi (cf. [Mor]) defines a value function which is a quasi-valuation satisfying a few more conditions. Given an integral Dubrovin valuation ring  $B$  of a central simple algebra  $S$ , Morandi shows that there is a value function  $w$  on  $S$  with  $B$  as its value ring (the value ring of  $w$  is defined as the set of all  $x \in S$  such that  $w(x) \geq 0$ ). Morandi also proves the converse, that if  $w$  is a value function on  $S$ , then the value ring is an integral Dubrovin valuation ring.

The use of value functions allow a number of results about invariant valuation rings to be extended to Dubrovin valuation rings. Also, their use has led to simpler and more natural proofs of a number of results on invariant valuation rings.

Tignol and Wadsworth (cf. [TW]) have developed a powerful theory which utilizes filtrations. They consider the notion of gauges, which are the surmultiplicative value functions for which the associated graded algebra is semisimple, and which also satisfy a defectlessness condition. The gauges are defined on finite dimensional semisimple algebras over valued fields with arbitrary value groups. Tignol and Wadsworth also show the relation between their value functions and Morandi's.

Quasi-valuations generalize both value functions and gauges (the axioms of quasi-valuations are contained in the axioms of value functions and gauges). Although we do not obtain Tignol and Wadsworth's decisive results concerning the Brauer group, we do get a workable theory, in which we are able to answer questions regarding the structure of rings using quasi-valuation theory. One difference with value functions, for example, is that on a field, Morandi's construction of a value function is automatically a valuation, while that is not the case for quasi-valuations. Gauges on a field reduce to exponential quasi-valuations, a special case of quasi-valuations. We believe that, even over a field, quasi-valuations are a natural generalization of valuations, and enrich valuation theory even for commutative algebras.

Whereas for a valuation on a field  $M$  is automatically a group since  $v(x^{-1}) = -v(x)$ , the situation is different with quasi-valuations (since  $w(x^{-1}) + w(x) \leq w(1)$ ). Thus it is more natural to map  $w$  to a monoid. In case  $w : R \rightarrow M \cup \{\infty\}$  is a quasi-valuation for  $M$  a group, we say that  $w$  is *group-valued* (rather than calling it a pseudo-valuation); we do not require  $w$  to be surjective.

Note that it does not follow from the axioms of a quasi-valuation that  $w(1)$  is necessarily 0. We shall see examples in which  $w(1) \neq 0$ .

We list here some of the common symbols we use for  $v$  a valuation on a field  $F$  and  $w$  a quasi-valuation on a ring  $R$  (usually we write  $E$  instead of  $R$ ):

$O_v = \{x \in F \mid v(x) \geq 0\}$ ; the valuation ring.

$I_v = \{x \in F \mid v(x) > 0\}$ ; the valuation ideal.

$O_w = \{x \in R \mid w(x) \geq 0\}$ ; the quasi-valuation ring.

$I_w = \{x \in R \mid w(x) > 0\}$ ; the quasi-valuation ideal.

$J_w$ ; the Jacobson radical of  $O_w$ .

$\Gamma_v$ ; the value group of the valuation  $v$ .

$M_w$ ; the value monoid of the quasi-valuation  $w$ , i.e., the submonoid of  $M$  generated by  $w(R \setminus \{0\})$ .

Note that  $w$  is group-valued iff  $M_w$  is cancellative, since any ordered abelian cancellative monoid has a group of fractions.

Here is a brief overview of this paper. In section 1 we present some general properties of quasi-valuations on rings as well as some basic examples. Thereafter we work under the assumption that  $F$  denotes a field with a valuation  $v$ ,  $E/F$  usually is a finite field extension, and  $w$  is a quasi-valuation on  $E$  such that  $w|_F = v$ . In section 2 we discuss some of the basic results regarding quasi-valuations and their corresponding rings; most of the results in this section are valid in the more general case where  $E$  is a finite dimensional  $F$ -algebra. In section 3 we prove that a quasi-valuation ring satisfies INC and LO over  $O_v$ ; in fact, LO is valid in the case where  $E$  is a finite dimensional  $F$ -algebra. In section 4 we introduce a notion called PIM (positive isolated monoid); the PIMs enable us to prove that a quasi-valuation ring satisfies GD over  $O_v$ . We also describe a connection between the closure of a PIM and the prime ideals of  $O_w$ . In sections 5 and 6 we assume that  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$ , the value group of the valuation. In section

5 we generalize some properties of valuation rings and we discuss the set of all expansions of a quasi-valuation ring (and one of his maximal ideals). We show that a quasi-valuation ring satisfies GU over  $O_v$ . We also construct a quasi-valuation that arises naturally from  $w$ ; this quasi-valuation is one of the key steps to give a bound on the size of the prime spectrum of the quasi-valuation ring. In section 6 we prove that any valuation whose valuation ring contains  $O_w$  dominates  $w$ . We also show that  $O_w$  satisfies the height formula (since we have the property GU). In section 7 we discuss exponential quasi-valuations; we do not assume that  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$ . Instead, we assume the weaker hypothesis that the value monoid is weakly cancellative. We prove that an exponential quasi-valuation must be of the form  $w = \min\{u_1, \dots, u_k\}$  for valuations  $u_i$  on  $E$  extending  $v$ . We obtain a 1:1 correspondence between  $\{\text{exponential quasi-valuations extending } v\}$  and  $\{\text{integrally closed quasi-valuation rings}\}$ . We also deduce that the number of exponential quasi-valuations is bounded by  $2^{[E:F]} - 1$ . In section 8 we construct a total ordering on a suitable amalgamation of  $M_w$  and  $\Gamma_{\text{div}}$  that allows us to compare elements of  $\Gamma_{\text{div}}$  with elements of  $M_w$ . Then we show that there exists a valuation  $u$  whose valuation ring contains  $O_w$  and  $u$  dominates  $w$ . In section 9 we review some of the notions of cuts and we present the construction of the cut monoid (of a totally ordered abelian group) which is an  $\mathbb{N}$ -strictly ordered abelian monoid. Then we introduce the filter quasi-valuations, which are quasi-valuations that can be defined on any  $O_v$ -algebra. The filter quasi-valuation is induced by an  $O_v$ -algebra and the valuation  $v$ ; its values lie inside the cut monoid of  $\Gamma_v$ . This gives us our main theorem, which enables us to apply the methods developed in the previous sections, as indicated below in Theorem 9.37. In section 10 we show that filter quasi-valuations and their cut monoids satisfy some properties which general quasi-valuations and their monoids do not necessarily share. This enables us to prove a stronger version of Theorem 8.15 when dealing with filter quasi-valuations. We also show that the filter quasi-valuation construction respects localization at prime ideals of  $O_v$ . Finally, we present the minimality of the filter quasi-valuation with respect to a natural partial order.

## §1 GENERAL QUASI-VALUATIONS AND EXAMPLES

In this section we present some of the basic definitions and properties regarding quasi-valuation on rings. We also present some examples of quasi-valuations. In particular, we give examples of quasi-valuations on integral domains which cannot be extended to their fields of fractions.

*Remark 1.1.* If the monoid  $M$  is cancellative then  $w(1) \leq 0$ .

*Proof.*  $w(1) = w(1^2) \geq w(1) + w(1)$ .

Here is a trivial example of a quasi-valuation on  $\mathbb{Z}$ .

*Example 1.2.*  $w(0) = \infty$  and  $w(z) = -1$  for all  $z \neq 0$ .

**Lemma 1.3.** *Let  $w$  be a quasi-valuation on a ring  $R$  and suppose that  $w(-1) = 0$ . Then  $w(a) = w(-a)$  for any  $a \in R$ .*

*Proof.* Let  $a \in R$ ; then

$$w(-a) = w(-1 \cdot a) \geq w(-1) + w(a) = w(a).$$

By symmetry  $w(a) = w(-(-a)) \geq w(-a)$ .

The following lemma generalizes a well known lemma from valuation theory, with the same proof. We prove it here for the reader's convenience.

**Lemma 1.4.** *Let  $w$  be a quasi-valuation on a ring  $R$  and suppose that  $w(-1) = 0$ .*

*Let  $a, b \in R$ ;*

*(i) If  $w(a) \neq w(b)$  then  $w(a + b) = \min\{w(a), w(b)\}$ .*

*(ii) If  $w(a + b) > w(a)$  then  $w(b) = w(a)$ .*

*Proof.* We prove the first statement; the second follows easily. Assume  $w(a) < w(b)$ ; then  $w(a + b) \geq w(a)$ . On the other hand,

$$w(a) = w(a + b - b) \geq \min\{w(a + b), w(b)\} = w(a + b).$$

Q.E.D.

In order to be able to use techniques from valuation theory, we define the following special elements with respect to a quasi-valuation.

*Definition 1.5.* Let  $w$  be a quasi-valuation on a commutative ring  $R$ . An element  $c \in R$  is called *stable* with respect to  $w$  if

$$w(cx) = w(c) + w(x)$$

for every  $x \in R$ .

**Lemma 1.6.** *Let  $w$  be a quasi-valuation on a commutative ring  $R$  such that  $w(1) = 0$ . Let  $c$  be an invertible element of  $R$ . Then  $c$  is stable iff  $w(c) = -w(c^{-1})$ .*

*Proof.*  $(\Rightarrow)$   $c$  is stable and invertible; thus

$$0 = w(1) = w(cc^{-1}) = w(c) + w(c^{-1}),$$

i.e.,  $w(c) = -w(c^{-1})$ .

$(\Leftarrow)$  Let  $x \in R$ ; we have

$$\begin{aligned} w(x) &= w(c^{-1}cx) \geq w(c^{-1}) + w(cx) \\ &\geq w(c^{-1}) + w(c) + w(x) = w(x). \end{aligned}$$

Hence equality holds, and since  $w(c^{-1}) = -w(c)$  is invertible, we have

$$w(cx) = w(c) + w(x).$$

Q.E.D.

*Definition 1.7.* An *exponential quasi-valuation*  $w$  on a ring  $R$  is a quasi-valuation such that for every  $x \in R$ ,

$$(1) \quad w(x^n) = nw(x), \quad \forall n \in \mathbb{N}.$$

We note that any exponential quasi-valuation  $w$  with  $M_w$  cancellative satisfies  $w(1) = w(-1) = 0$ .

Usually we need only check the following special case of (1):

**Proposition 1.8.** *Let  $R$  be a ring and let  $M$  be a totally ordered cancellative abelian monoid. Let  $w : R \rightarrow M \cup \{\infty\}$  be a quasi-valuation satisfying the condition: for every  $x \in R$ ,  $w(x^2) = 2w(x)$ . Then  $w$  is an exponential quasi-valuation.*

*Proof.* It is obvious that  $w(x^n) = nw(x)$  for  $n = 1, 2$ . Assume by induction that the statement is true for every  $k < n$ .

If  $n$  is even,

$$w(x^n) = w((x^{n/2})^2) = 2w(x^{n/2}) = 2(n/2)w(x) = nw(x).$$

If  $n$  is odd,

$$w(x^n) \geq nw(x)$$

and

$$\begin{aligned} (n+1)w(x) &= w(x^{n+1}) = w(x^n x) \\ &\geq w(x^n) + w(x) \geq nw(x) + w(x) = (n+1)w(x). \end{aligned}$$

Therefore equality holds and cancelling  $w(x)$ , we get  $w(x^n) = nw(x)$ .  
Q.E.D.

We present here some examples of quasi-valuations:

*Example 1.9.* Let  $R$  be a UFD, let  $F$  be its field of fractions, and fix a non-zero  $n \in R$  (not necessarily prime). For any  $0 \neq \frac{c}{d} \in F$  there exists a unique  $e \in R$  such that  $\frac{c}{d} = n^e \frac{a}{b}$ , where  $a, b \in R$ ,  $n \nmid a$ ,  $(n, b) = 1$  and  $(a, b) = 1$ . Define  $w_n(\frac{c}{d}) = e$  and  $w_n(0) = \infty$ .  $w_n$  is called the  $n$ -adic quasi-valuation on  $F$ . A detailed analysis of the  $n$ -adic quasi-valuation on  $\mathbb{Q}$  is given in [Ste, ch. 16].

Note that whenever

$$n = \prod_{i=1}^r p_i$$

where the  $p_i$ 's are distinct non-zero primes, the  $n$ -adic quasi-valuation  $w_n$  is equal to

$$\min_{1 \leq i \leq r} \{w_{p_i}\};$$

where each  $w_{p_i}$  is a valuation.

*Example 1.10.* Let  $R$  be a ring and let  $\{w_i\}_{i \in I}$  be a set of quasi-valuations having the same value monoid (i.e., for every  $i, j \in I$ ,  $M_{w_i} = M_{w_j} =: M$ ). Assume that for any  $r \in R$ ,  $\{w_i(r) \mid i \in I\}$  is finite. This is automatic when  $I$  is finite but also holds for an arbitrary set of valuations over a number field (i.e., the  $w_i$ 's are valuations) since for every  $r$ ,  $w_i(r) = 0$  for almost all  $i$ . Define

$$u = \min_{i \in I} \{w_i\}.$$

Then  $u$  is a quasi-valuation with values inside the monoid  $M$ .

Example 1.10 will be generalized and become a tool for computing Krull dimension in section 5.

Let  $R$  be an integral domain and assume that  $v$  is a valuation on  $R$ . Then  $v$  is automatically a valuation on  $F$ , the field of fractions of  $R$ . However, the

situation is different in the quasi-valuation case. A quasi-valuation on an integral domain cannot always be extended to its field of fractions. For example, let  $v$  be a valuation on a field  $F$  with value group  $\Gamma$  and let  $\alpha \in \Gamma$  be a positive element. Define  $w_\alpha : F \rightarrow \Gamma \cup \{\infty\}$  by

$$w_\alpha(x) = \begin{cases} v(x) & \text{if } v(x) < \alpha \\ \infty & \text{otherwise.} \end{cases}$$

$w_\alpha$  is a quasi-valuation on  $O_v$  but obviously cannot be extended to  $F$  (since  $w^{-1}(\{\infty\})$  is an ideal of  $F$ ).

We shall now present an example of a quasi-valuation on  $\mathbb{Z}$  with value group  $\Gamma = \mathbb{Z}$  such that  $w(x) \neq \infty$  for  $x \neq 0$  and for which the quasi-valuation cannot be extended to a quasi-valuation on its field of fractions  $\mathbb{Q}$ , such that the value monoid is torsion over  $\Gamma$ .

*Example 1.11.* Let  $v$  be any p-adic valuation on  $\mathbb{Z}$ ; we define a function  $w : \mathbb{Z} \rightarrow \mathbb{Z} \cup \{\infty\}$  by  $w(x) = [v(x)]^2$  (with the convention  $\infty^2 = \infty$ ). Note that  $\text{im}(w) \subseteq \mathbb{N} \cup \{0, \infty\}$ ; it is not difficult to see that  $w$  is a quasi-valuation on  $\mathbb{Z}$ . Indeed,  $w(0) = \infty$ ; if  $x, y \in \mathbb{Z}$ , then

$$\begin{aligned} w(xy) &= [v(xy)]^2 = [v(x) + v(y)]^2 \\ &= [v(x)]^2 + 2v(x)v(y) + [v(y)]^2 \geq w(x) + w(y). \end{aligned}$$

Finally, if  $x, y \in \mathbb{Z}$ , then

$$\begin{aligned} w(x+y) &= [v(x+y)]^2 \geq [\min\{v(x), v(y)\}]^2 \\ &= \min\{[v(x)]^2, [v(y)]^2\} = \min\{w(x), w(y)\}. \end{aligned}$$

Now, let  $x, y$  be non-zero integers such that  $w(x), w(y) > 0$  and let  $n \in \mathbb{N}$ ; we have

$$\begin{aligned} w(y^n) &= w(y^n x \frac{1}{x}) \geq w(y^n x) + w(\frac{1}{x}) \\ &= [v(y^n x)]^2 + w(\frac{1}{x}) = [v(y^n) + v(x)]^2 + w(\frac{1}{x}) \\ &= [v(y^n)]^2 + 2v(y^n)v(x) + [v(x)]^2 + w(\frac{1}{x}) \\ &= w(y^n) + 2v(y^n)v(x) + w(x) + w(\frac{1}{x}). \end{aligned}$$

Thus, cancelling  $w(y^n)$ , we get  $w(\frac{1}{x}) \leq -2v(y^n)v(x) - w(x)$  for every  $n \in \mathbb{N}$ . Hence  $w(\frac{1}{x})$  is less than every  $z \in \mathbb{Z}$ . Thus there cannot exist an  $n \in \mathbb{N}$  such that  $nw(\frac{1}{x}) \in \mathbb{Z}$ . Therefore  $w$  cannot be extended to  $\mathbb{Q}$  with value monoid torsion over  $\Gamma$ .

Let us consider an example of such a quasi-valuation extending a valuation in an infinite dimensional extension.

*Example 1.12.* Let  $v$  denote the trivial valuation on  $\mathbb{Q}$  and let  $w$  denote the  $\lambda^2$ -adic quasi-valuation (see Example 1.9) on  $\mathbb{Q}[\lambda]$ , where  $\lambda$  is a commutative indeterminate.

$w$  is a quasi-valuation which is not a valuation since  $w(\lambda^2) = 1 > 0 = 2w(\lambda)$  and obviously  $w$  extends  $v$ .

In order to avoid pathological cases, we shall study quasi-valuations extending a given valuation on a field.

In this paper,  $F$  denotes a field with a valuation  $v$ ,  $E/F$  usually is a finite field extension with  $n = [E : F]$ , and  $w$  is a quasi-valuation on  $E$  such that  $w|_F = v$ . We shall see that the theory is surprisingly rich in these cases, especially for exponential quasi-valuations.

Some of the results are valid in the more general case where  $F$  is a field and  $E$  is a finite dimensional  $F$ -algebra. When possible, we shall discuss this more general scope.

## §2 BASIC PROPERTIES AND EXAMPLES

In this section, until Example 2.7, we consider the more general case for which  $E$  is a (not necessarily commutative) finite dimensional  $F$ -algebra.

Note that  $\Gamma_v$  embeds in  $M_w$  since  $w$  extends  $v$ . Also note that since  $w$  extends  $v$  and  $v$  is a valuation, then by Lemma 1.6 every nonzero  $x \in F$  is stable with respect to  $w$ ; 0 is obviously stable. So, every  $x \in F$  is stable with respect to  $w$ . Furthermore,  $w(-1) = v(-1) = 0$ . Hence, applying induction to Lemma 1.4, for any  $x_1, x_2, \dots, x_n$  such that there exists a single  $k$  satisfying  $w(x_k) = \min_{1 \leq i \leq n} \{w(x_i)\}$ , we have

$$w\left(\sum_{i=1}^n x_i\right) = w(x_k).$$

**Lemma 2.1.** *The elements of  $w(E \setminus \{0\})$  lie in  $\leq n$  cosets over  $\Gamma_v$ , where we recall  $[E : F] = n$ .*

*Proof.* Let  $b_1, \dots, b_m \in E \setminus \{0\}$  such that  $w(b_i) + \Gamma_v \neq w(b_j) + \Gamma_v$  for every  $i \neq j$ . We show that the set  $\{b_k\}_{k=1}^m$  is linearly independent. Assume to the contrary, that there exist  $\alpha_k \in F$  not all 0 such that  $\sum \alpha_i b_i = 0$ . Note that  $w(\alpha_i b_i) = w(\alpha_i) + w(b_i)$  and therefore the value of each  $\alpha_i b_i$  lies in a different coset; in particular,  $w(\alpha_i b_i) \neq w(\alpha_j b_j)$  for every  $i \neq j$ . Hence

$$w\left(\sum \alpha_i b_i\right) = \min_{1 \leq i \leq m} \{w(\alpha_i b_i)\} \neq w(0),$$

a contradiction.

Q.E.D.

*Remark 2.2.* Let  $O_v$  be a valuation ring of  $F$  and let  $E$  be an  $F$ -algebra with  $[E : F] = n$ . Then every set  $\{b_1, b_2, \dots, b_m\} \subseteq E$  with  $m > n$  satisfies the following: there exist  $\alpha_i \in O_v$  such that  $\sum_{i=1}^m \alpha_i b_i = 0$  and  $\alpha_{i_0} = 1$  for some  $1 \leq i_0 \leq m$ , i.e.,

$$b_{i_0} = - \sum_{i \neq i_0} \alpha_i b_i.$$

*Proof.* The  $b_i$  are linearly dependent over  $F$ , so there exist  $\beta_i \in F$  not all 0 such that  $\sum_{i=1}^m \beta_i b_i = 0$ . We choose a  $\beta_{i_0}$  such that  $v(\beta_{i_0}) \leq v(\beta_i)$  for all  $1 \leq i \leq m$  and multiply by  $\beta_{i_0}^{-1}$ . Writing  $\alpha_i = \beta_{i_0}^{-1} \beta_i$ , we have  $\alpha_i \in O_v$  and  $\alpha_{i_0} = 1$ .

Q.E.D.



**Lemma 2.3.** *Let  $R \supseteq O_v$  be a ring contained in  $E$ . Then every f.g.  $O_v$ -submodule of  $R$  is spanned by  $\leq n$  generators.*

*Proof.* By Remark 2.2 every set  $\{b_1, b_2, \dots, b_{n+1}\}$  of  $n+1$  elements can be reduced to a set of  $n$  elements which span the same submodule.

**Corollary 2.4.** *Every f.g. ideal of  $O_w$  is generated by  $\leq n$  generators.*

Note that if an ideal  $I$  is not f.g., one cannot argue that  $I$  is generated by  $\leq n$  generators. Indeed, let  $v$  be a non-discrete valuation; thus  $O_v$  itself is not Noetherian. Now, take  $E = F$  and  $w = v$ .

**Lemma 2.5.**  $[O_w/I_v O_w : O_v/I_v] \leq n$ .

*Proof.* Let  $\overline{a_1}, \dots, \overline{a_m} \in O_w/I_v O_w$  be linearly independent over  $O_v/I_v$ . Let  $a_1, \dots, a_m$  be their representatives. We prove that  $a_1, \dots, a_m$  are linearly independent over  $F$ . Assume to the contrary and apply Remark 2.2 to get  $\sum_{i=1}^m \alpha_i a_i = 0$  where  $\alpha_i \in O_v$  and  $\alpha_{i_0} = 1$  for some  $1 \leq i_0 \leq m$ . So, we have  $\sum_{i=1}^m \overline{\alpha_i a_i} = 0$  where  $\overline{\alpha_{i_0}} = \overline{1}$ . This contradicts the linear independence of the  $\overline{a_i}$ 's.

Q.E.D.

**Corollary 2.6.**  $[O_w/I_w : O_v/I_v] \leq n$

*Proof.* Note that  $I_w \supseteq I_v O_w$  and thus we have the natural epimorphism

$$O_w/I_v O_w \twoheadrightarrow O_w/I_w.$$

The result now follows from Lemma 2.5.

Q.E.D.

*Example 2.7.* Let  $F$  be a field with valuation  $v$ , and let  $E$  be a finite dimensional  $F$ -algebra with basis  $B = \{b_1 = 1, b_2, \dots, b_n\}$ . Assume that  $B$  satisfies the following: for every  $1 \leq i, j \leq n$ ,  $b_i b_j = \delta b_k$  for some  $1 \leq k \leq n$  and  $\delta \in O_v$  (in other words the basis is projectively multiplicative with respect to  $O_v$ ). Define

$$w\left(\sum_{i=1}^n \alpha_i b_i\right) = \min_{1 \leq i \leq n} \{v(\alpha_i)\}.$$

It is obvious that  $w$  extends  $v$ ; we show that  $w$  is a quasi-valuation. Indeed, assuming

$$v(\alpha_{i_0}) = w\left(\sum_{i=1}^n \alpha_i b_i\right) \leq w\left(\sum_{i=1}^n \beta_i b_i\right) = v(\beta_{i_1}),$$

we have

$$\begin{aligned} w\left(\sum_{i=1}^n \alpha_i b_i + \sum_{i=1}^n \beta_i b_i\right) &= w\left(\sum_{i=1}^n (\alpha_i + \beta_i) b_i\right) \\ &= \min_{1 \leq i \leq n} \{v(\alpha_i + \beta_i)\} \geq \min_{1 \leq i \leq n} \{\min\{v(\alpha_i), v(\beta_i)\}\} \\ &= v(\alpha_{i_0}) = w\left(\sum_{i=1}^n \alpha_i b_i\right). \end{aligned}$$

Now,

$$w\left(\sum_{i=1}^n \alpha_i b_i \sum_{j=1}^n \beta_j b_j\right) = w\left(\sum_{i,j} \alpha_i \beta_j b_i b_j\right) = w\left(\sum_{i,j} \alpha_i \beta_j \delta_{i,j} b_{i,j}\right),$$

for  $\delta_{i,j} \in O_v$  and  $b_{i,j} \in B$ . By the previous calculation, we have

$$\begin{aligned} w\left(\sum_{i,j} \alpha_i \beta_j \delta_{i,j} b_{i,j}\right) &\geq \min_{i,j} \{w(\alpha_i \beta_j \delta_{i,j} b_{i,j})\} = \min_{i,j} \{v(\alpha_i \beta_j \delta_{i,j})\} \\ &\geq \min_{i,j} \{v(\alpha_i \beta_j)\} = \min_{i,j} \{v(\alpha_i) + v(\beta_j)\} = \min_i \{v(\alpha_i)\} + \min_j \{v(\beta_j)\} = \\ &w\left(\sum_{i=1}^n \alpha_i b_i\right) + w\left(\sum_{j=1}^n \beta_j b_j\right). \end{aligned}$$

**Lemma 2.8.** *Let  $E/F$  be a finite field extension and let  $w$  be a quasi-valuation on  $E$  extending a valuation  $v$  on  $F$ . Then  $w(x) \neq \infty$  for all  $0 \neq x \in E$ .*

*Proof.* If  $w(x) > 0$ , writing  $\sum_{i=0}^n \alpha_i x^i = 0$  for  $\alpha_i \in O_v$  and  $\alpha_0 \neq 0$ , we have

$$w(x) \leq w(\alpha_0) = v(\alpha_0) < \infty.$$

Q.E.D.

The following example will be used to provide negative answers to several natural questions.

*Example 2.9.* Let  $v$  be a valuation on a field  $F$  and let  $(M, +)$  be a totally ordered abelian monoid containing  $\Gamma_v = v(F \setminus \{0\})$ . Let  $E = F[e]$  be a Kummer field extension with  $e \notin F$  and  $e^2 \in O_v$ . Define  $w$  on  $E$  in the following way:

$$w(a + be) = \min\{v(a), v(b) - \gamma\}$$

for some invertible element  $\gamma \in M$ ,  $\gamma \geq 0$ . We shall prove that  $w$  is indeed a quasi-valuation extending  $v$ : assume  $w(a + be) \leq w(c + de)$ ; then

$$\begin{aligned} w(a + be + c + de) &= \min\{v(a + c), v(b + d) - \gamma\} \\ &\geq \min\{\min\{v(a), v(c)\}, \min\{v(b), v(d)\} - \gamma\} \\ &= \min\{\min\{v(a), v(c)\}, \min\{v(b) - \gamma, v(d) - \gamma\}\} \\ &= \min\{v(a), v(b) - \gamma\} = w(a + be). \end{aligned}$$

Next,

$$\begin{aligned} w((a + be)(c + de)) &= \min\{v(ac + bde^2), v(ad + bc) - \gamma\} \\ &\geq \min\{\min\{v(ac), v(bde^2)\}, \min\{v(ad), v(bc)\} - \gamma\} \\ &= \min\{\min\{v(a) + v(c), v(b) + v(d) + v(e^2)\}, \min\{v(a) + v(d) - \gamma, v(b) + v(c) - \gamma\}\} \\ &= \min\{v(a) + v(c), v(b) + v(d) + v(e^2), v(a) + v(d) - \gamma, v(b) + v(c) - \gamma\} \end{aligned}$$

$$\geq \min\{v(a) + v(c), v(b) + v(d), v(a) + v(d) - \gamma, v(b) + v(c) - \gamma\}.$$

Note that

$$(2) \quad w(a + be) + w(c + de) = \min\{v(a), v(b) - \gamma\} + \min\{v(c), v(d) - \gamma\}.$$

So, if the sum on the right side of (2) is  $v(a) + v(c)$ , then

$$v(a) + v(c) \leq \min\{v(a) + v(d) - \gamma, v(b) + v(c) - \gamma, v(b) - \gamma + v(d) - \gamma\}$$

and thus  $\leq v(b) + v(d)$ . We note that if  $\gamma > 0$  then in this case we have an equality  $w((a + be)(c + de)) = w(a + be) + w(c + de)$ . If the sum on the right side of (2) is  $v(a) + v(d) - \gamma$  or  $v(b) - \gamma + v(c)$ , we have a similar situation and again an equality (if  $\gamma > 0$ ). If the sum on the right side of (2) is  $v(b) - \gamma + v(d) - \gamma$ , then

$$v(b) - \gamma + v(d) - \gamma \leq \min\{v(a) + v(c), v(b) + v(d), v(a) + v(d) - \gamma, v(b) + v(c) - \gamma\}.$$

So, we have  $w((a + be)(c + de)) \geq w(a + be) + w(c + de)$ .

Note that if  $\gamma \in \Gamma_v$  then taking  $c \in F$  such that  $v(c) = \gamma$ , we have  $O_w = O_v[ce]$ . If  $\gamma \notin \Gamma_v$  then

$$O_w = \bigcup_{c \in F, v(c) > \gamma} O_v[ce].$$

For example if one takes  $F = \mathbb{Q}$ ,  $v = v_p$ ,  $M = \mathbb{Z}$  and  $\gamma = n \in \mathbb{N} \cup \{0\}$ ; then  $O_w = \mathbb{Z}_p[p^n e]$ .

### §3 INC, LO AND K-DIM

For any ring  $R$ , we denote by  $\text{K-dim} R$  the classical Krull dimension of  $R$ , by which we mean the maximal length of the chains of prime ideals of  $R$ . Our next goal is to prove that  $O_w$  satisfies INC (incomparability) and LO (lying over) over  $O_v$ ; cf. [Row, p. 184-192].

For the reader's convenience we review these notions here. Let  $C \subseteq R$  be rings. We say that  $R$  satisfies INC over  $C$  if whenever  $Q_0 \subset Q_1$  in  $\text{Spec}(R)$  we have  $Q_0 \cap C \subset Q_1 \cap C$ . We say that  $R$  satisfies LO over  $C$  if for any  $P \in \text{Spec}(C)$  there is  $Q \in \text{Spec}(R)$  lying over  $P$ , i.e.,  $Q \cap C = P$ .

*Remark 3.1.* Let  $v$  be a valuation with valuation ring  $O_v$  and let  $I$  be an ideal of  $O_v$ . If  $x \in I$  and  $y \in O_v$  such that  $v(y) \geq v(x)$ , then  $y \in I$ . Indeed,  $O_v y \subseteq O_v x \subseteq I$ .

*Remark 3.2.* Let  $C \subseteq R$  be rings. Let  $I_1 \subseteq I_2$  be ideals of  $R$  such that  $I_1 \cap C = I_2 \cap C$ . If  $x \in I_2$  and  $\sum_{i=0}^m \alpha_i x^i \in I_2$  for  $\alpha_i \in C$ , then  $\alpha_0 \in I_1$ . Indeed,  $\sum_{i=1}^m \alpha_i x^i \in I_2$  (since  $x \in I_2$ ), so  $\alpha_0 \in I_2$ . Therefore  $\alpha_0 \in I_1$  (since  $\alpha_0 \in I_2 \cap C = I_1 \cap C$ ).

**Lemma 3.3.** *Let  $C \subseteq R$  be commutative rings. Let  $Q_1 \subseteq Q_2$  be prime ideals of  $R$  with  $Q_1 \cap C = Q_2 \cap C$ . If  $\sum_{i=0}^m \alpha_i x^i \in Q_1$  for  $\alpha_i \in C$  and  $x \in Q_2 \setminus Q_1$ ; then each  $\alpha_i \in Q_1$ .*

*Proof.* By Remark 3.2,  $\alpha_0 \in Q_1$ . Note that  $x \sum_{i=1}^m \alpha_i x^{i-1} \in Q_1$  and since  $x \notin Q_1$  we have  $\sum_{i=1}^m \alpha_i x^{i-1} \in Q_1$ . Again, we apply Remark 3.2 and get  $\alpha_1 \in Q_1$ . We continue by induction and get each  $\alpha_i \in Q_1$ .

Q.E.D.

**Definition 3.4.** Let  $C \subseteq R$  be rings. We call  $R$  *quasi-integral* over  $C$  if for every  $r \in R$  there exists a polynomial  $\sum c_i r^i = 0$  with  $c_i \in C$  and  $c_{i_0} = 1$  for some  $i_0$  (depending on  $r$ ).

**Remark 3.5.** Recall that  $E$  is a finite field extension of a field  $F$  with valuation  $v$  and valuation ring  $O_v$ . Note that for any  $E \supseteq R \supseteq O_v$ ,  $R$  is quasi-integral over  $O_v$ , by Remark 2.2 applied to  $\{1, r, \dots, r^n\}$ . In particular, the quasi-valuation ring  $O_w$  is quasi-integral over  $O_v$ .

**Remark 3.6.** If  $R$  is quasi-integral over  $C$  and  $Q \triangleleft R$ , then  $R/Q$  is quasi-integral over  $C/C \cap Q$ .

*Proof.* For any  $r + Q \in R/Q$ , there exists a polynomial  $\sum_{i=0}^n c_i r^i = 0$  for  $c_i \in C$  and  $c_{i_0} = 1$  for some  $i_0$ . We pass to the image in  $R/Q$  over  $(C + Q)/Q \approx C/C \cap Q$ .  
Q.E.D.

**Theorem 3.7.** Any commutative quasi-integral ring extension  $R$  of  $O_v$  satisfies INC over  $O_v$ .

*Proof.* Let  $Q_1 \subset Q_2$  be prime ideals of  $R$ . Assume to the contrary, that  $Q_1 \cap O_v = Q_2 \cap O_v$ . Let  $x \in Q_2 \setminus Q_1$  and by assumption write  $\sum_{i=0}^n \alpha_i x^i = 0$  for  $\alpha_i \in O_v$ ,  $\alpha_{i_0} = 1$  for some  $i_0$ . By Lemma 3.3,  $\alpha_{i_0} \in Q_1$ , a contradiction.  
Q.E.D.

**Corollary 3.8.** If  $R$  is commutative and quasi-integral over  $O_v$ , then  $K\text{-dim} R \leq K\text{-dim} O_v$ .

*Proof.* By Theorem 3.7  $R$  satisfies INC over  $O_v$ ; thus  $K\text{-dim} R \leq K\text{-dim} O_v$ .  
Q.E.D.

**Corollary 3.9.**  $K\text{-dim} O_w \leq K\text{-dim} O_v$ , for any quasi-valuation  $w$  extending  $v$  on a finite dimensional field extension  $E$  of  $F$ .

*Proof.*  $O_w$  is quasi-integral over  $O_v$  by Remark 3.5; now, use Corollary 3.8.  
Q.E.D.

The following three lemmas are valid in the more general case where  $E$  is a finite dimensional  $F$ -algebra.

**Lemma 3.10.** Let  $I$  be a proper ideal of  $O_v$ ; then every  $x \in IO_w$  can be written as  $x = ar$  for  $a \in I$ ,  $r \in O_w$ .

*Proof.* Every  $x \in IO_w$  is of the form  $\sum_{i=1}^n a_i r_i$  for  $a_i \in I$ ,  $r_i \in O_w$ ; so take  $a = a_{i_0}$  with minimal  $v$ -value and write  $x = ar$  for appropriate  $r \in O_w$ .  
Q.E.D.

**Lemma 3.11.** Let  $I$  be a proper ideal of  $O_v$ ; then  $IO_w$  is a proper ideal of  $O_w$  and  $IO_w \cap O_v = I$ .

*Proof.* Let  $x \in IO_w$ ; by Lemma 3.10 we may write  $x = ar$  for  $a \in I$  and  $r \in O_w$ . Thus, since  $w(ar) \geq w(a) = v(a) > 0$ ,  $1 \notin IO_w$  and  $IO_w$  is a proper ideal of  $O_w$ . Now, let  $x \in IO_w \cap O_v$  and write  $x = ar$  for  $a \in I$  and  $r \in O_w$ . Then  $r \in F \cap O_w$  and thus  $r \in O_v$ ; hence  $IO_w \cap O_v = I$ .  
Q.E.D.

**Lemma 3.12.**  $O_w$  satisfies LO over  $O_v$ .

*Proof.* Let  $P \in \text{Spec}(O_v)$  and denote  $T = \{A \triangleleft O_w \mid A \cap O_v = P\}$ . By Lemma 3.11,  $PO_w$  is a proper ideal of  $O_w$  lying over  $P$ . Thus  $T \neq \emptyset$ . Now,  $T$  with the partial order of containment satisfies the conditions of Zorn's Lemma and thus there exists  $Q \triangleleft O_w$  lying over  $P$ , maximal with respect to containment. We shall prove that  $Q \in \text{Spec}(O_w)$ . Indeed, let  $A, B$  be ideals in  $O_w$  with  $AB \subseteq Q$  and assume that  $A, B \not\subseteq Q$ . Then

$$(Q + A) \cap O_v \neq P \quad \text{and} \quad (Q + B) \cap O_v \neq P;$$

so there exist  $a \in A$ ,  $b \in B$ ,  $q, q' \in Q$  such that  $q + a, q' + b \in O_v \setminus P$ . However

$$(q + a)(q' + b) = qq' + qb + aq' + ab \in Q \cap O_v = P,$$

a contradiction.

Q.E.D

#### §4 PIMS AND GD

In this section we introduce the notion of PIM. We show that the convex hull of  $H^{\geq 0}$  in  $M$ , where  $H$  is an isolated subgroup of  $\Gamma$ , is a PIM. We use the closures of these PIMs to prove that  $O_w$  satisfies GD over  $O_v$ . We also obtain a connection between the height of a prime ideal  $P$  of  $O_v$  and the height of a prime ideal of  $O_w$  lying over  $P$ . Finally, we present a connection between the closure of a PIM and the prime ideals of  $O_w$ .

From now through Corollary 4.10 we consider abstract properties of a totally ordered abelian monoid  $M$  containing a group  $\Gamma$ .

Let  $N \subseteq M$ , we denote

$$N^{\geq 0} = \{m \in N \mid m \geq 0\}$$

and

$$N^{< 0} = \{\alpha \in N \mid \alpha < 0\}$$

In particular,  $M^{\geq 0} = \{m \in M \mid m \geq 0\}$ . Note that  $M^{\geq 0}$  is a submonoid of  $M$ .

*Definition 4.1.* A subset  $N \subseteq M^{\geq 0}$  is called a *positive isolated monoid* (PIM) if  $N$  is a submonoid of  $M^{\geq 0}$  and for every  $\gamma \in N$  we have

$$\{\delta \in M^{\geq 0} \mid \delta \leq \gamma\} \subseteq N.$$

Recall that an isolated subgroup of a totally ordered abelian group  $\Gamma$  is a subgroup  $H$  of  $\Gamma$  such that  $\{\gamma \in \Gamma \mid 0 \leq \gamma \leq h\} \subseteq H$  for any  $h \in H$  (some texts call such subgroups "convex" or "distinguished"); see [End, p. 47].

For any totally ordered set  $A$  and a subset  $B \subseteq A$  the convex hull of  $B$  in  $A$  is defined by

$$\text{hull}_A(B) = \{a \in A \mid \exists b_1, b_2 \in B \text{ such that } b_1 \leq a \leq b_2\}.$$

Let  $H$  be an isolated subgroup of  $\Gamma$ ; it is not difficult to see that

$$\text{hull}_M(H^{\geq 0}) = [\text{hull}_M(H)]^{\geq 0}.$$

*Remark 4.2.* Let  $H \leq \Gamma$  be an isolated subgroup.  $\text{hull}_M(H^{\geq 0})$  is a PIM and  $\text{hull}_M(H^{\geq 0}) \cap \Gamma = H^{\geq 0}$ . We prove that  $\text{hull}_M(H^{\geq 0})$  is indeed a PIM. Let  $m \in \text{hull}_M(H^{\geq 0})$  and  $0 \leq m' \leq m$ . Then there exists  $\alpha \in H^{\geq 0}$  such that  $0 \leq m' \leq m \leq \alpha$ ; thus  $m' \in \text{hull}_M(H^{\geq 0})$ .  $\text{hull}_M(H^{\geq 0})$  is a monoid since  $H^{\geq 0}$  is closed under addition.

*Remark 4.3.* The PIMs of  $M^{\geq 0}$  are linearly ordered by inclusion.

*Proof.* Assume  $N_1 \neq N_2$  and assume there exists  $\gamma \in N_1 \setminus N_2$ . Then for any  $\delta \in N_2$  we have  $\gamma > \delta$  (since otherwise  $\gamma \in N_2$ ), implying  $\delta \in N_1$  by assumption that  $N_1$  is isolated. Thus  $N_2 \subseteq N_1$ .

*Remark 4.4.* The correspondence  $H^{\geq 0} \rightarrow \text{hull}_M(H^{\geq 0})$  from

$$\{\text{positive part of isolated subgroups of } \Gamma\} \rightarrow \{\text{PIMs of } M^{\geq 0}\}$$

is an injection.

*Proof.* Recall that the isolated subgroups are linearly ordered, and thus their positive parts are also totally ordered. Assume that  $(H_1)^{\geq 0} \supset (H_2)^{\geq 0}$  and let  $\alpha \in (H_1)^{\geq 0} \setminus (H_2)^{\geq 0}$ . Then

$$\alpha \in \text{hull}_M((H_1)^{\geq 0}) \setminus \text{hull}_M((H_2)^{\geq 0}).$$

Q.E.D.

*Definition 4.5.* A PIM  $N \subseteq M^{\geq 0}$  is said to be *lie over* the positive part  $H^{\geq 0}$  of an isolated subgroup  $H$  if  $N \cap \Gamma = H^{\geq 0}$ .

*Example 4.6.* Let  $N$  be a PIM of  $M^{\geq 0}$ . Then

$$H = (N \cap \Gamma) \cup \{-\alpha \mid \alpha \in N \cap \Gamma\}$$

is an isolated subgroup of  $\Gamma$ , and  $N$  lies over  $H^{\geq 0}$ .

*Remark 4.7.*  $\text{hull}_M(H^{\geq 0})$  is the minimal PIM lying over  $H^{\geq 0}$ ; i.e., for any PIM  $N \subset \text{hull}_M(H^{\geq 0})$  we have  $N \cap \Gamma \subset H^{\geq 0}$ .

*Proof.* Take  $\gamma \in \text{hull}_M(H^{\geq 0}) \setminus N$ ; then  $0 \leq \gamma \leq \alpha$  for some  $\alpha \in H^{\geq 0}$ . Assuming  $N$  lies over  $H^{\geq 0}$  we get  $\alpha \in N$  and thus  $\gamma \in N$ , a contradiction.

Q.E.D.

*Definition 4.8.* Let  $H \leq \Gamma$  be an isolated subgroup and let  $\text{hull}_M(H^{\geq 0})$  be its corresponding PIM. The *closure* of  $\text{hull}_M(H^{\geq 0})$ , denoted  $\overline{\text{hull}_M(H^{\geq 0})}$ , is the set

$$\bigcup \{N_i \mid N_i \text{ is a PIM lying over } H^{\geq 0}\}.$$

**Lemma 4.9.** *Notation as in Definition 4.8.  $\overline{\text{hull}_M(H^{\geq 0})}$  is a PIM of  $M^{\geq 0}$  lying over  $H^{\geq 0}$ .*

*Proof.* It is clear that  $\overline{\text{hull}_M(H^{\geq 0})}$  is a PIM and that  $H^{\geq 0} \subseteq \overline{\text{hull}_M(H^{\geq 0})}$ . On the other hand, if  $\alpha \in \overline{\text{hull}_M(H^{\geq 0})} \cap \Gamma$  then  $\alpha \in N_i \cap \Gamma$  for some  $N_i$  lying over  $H^{\geq 0}$ . Thus  $\alpha \in H^{\geq 0}$ .

Q.E.D.

**Corollary 4.10.**  $\overline{\text{hull}_M(H^{\geq 0})}$  is the maximal PIM lying over  $H^{\geq 0}$ .

We note at this point that for an isolated subgroup  $H$  of  $\Gamma$  and  $m \in M^{\geq 0}$ ,  $m \notin \text{hull}_M(H^{\geq 0})$  iff  $m > \alpha$  for every  $\alpha \in H^{\geq 0}$ . However, one can have  $m > \alpha$  for every  $\alpha \in H^{\geq 0}$  and still  $m \in \overline{\text{hull}_M(H^{\geq 0})}$ . We shall see in Lemma 10.6 an example of such an element (we denote it by  $\mathcal{C}$ ), where the monoid is the cut monoid. Examples 10.7 and 10.8 also provide examples of such elements. We shall discuss more about this fact after these examples.

#### GOING DOWN

We shall now use the notion of PIM to prove Going Down (GD) for quasi-valuation rings over valuation rings. Recall (see [Row, p. 191]) that in case  $C \subseteq R$  are rings, we say that  $R$  satisfies GD (going down) over  $C$  if for any  $P_0 \subseteq P_1 \in \text{Spec}(C)$  and every  $Q_1 \in \text{Spec}(R)$  lying over  $P_1$  there is  $Q_0 \subseteq Q_1$  in  $\text{Spec}(R)$  lying over  $P_0$ .

As usual,  $v$  is a valuation on a field  $F$ ,  $O_v$  its valuation ring,  $E$  is a finite field extension of  $F$  and  $w$  is a quasi-valuation on  $E$  extending  $v$  with quasi-valuation ring  $O_w$  and value monoid  $M_w$ .

Recall that there is a one to one correspondence between the set of all prime ideals of a valuation ring and the set  $G(\Gamma_v)$  of all isolated subgroups of  $\Gamma_v$  given by  $P \mapsto \{\alpha \in \Gamma_v \mid \alpha \neq v(p) \text{ and } \alpha \neq -v(p) \text{ for all } p \in P\}$ . (See [End. p. 47]).

**Lemma 4.11.** *Let  $P \in \text{Spec}(O_v)$ ,  $Q \in \text{Spec}(O_w)$  such that  $Q \cap O_v = P$  and  $x \in O_w \setminus Q$ . Let  $H$  denote the isolated subgroup corresponding to  $P$  and let  $\text{hull}_{M_w}(H^{\geq 0})$  denote the corresponding PIM. Then*

$$w(x) \in \overline{\text{hull}_{M_w}(H^{\geq 0})}.$$

*Proof.* Assume to the contrary that  $w(x) \notin \overline{\text{hull}_{M_w}(H^{\geq 0})}$ . Let

$$N = \{m \in (M_w)^{\geq 0} \mid m \leq nw(x) \text{ for some } n \in \mathbb{N}\}.$$

$N$  is clearly a PIM strictly containing  $\overline{\text{hull}_{M_w}(H^{\geq 0})}$ ; thus  $N \cap \Gamma_v \supset H^{\geq 0}$ . Let  $\alpha \in (N \cap \Gamma_v) \setminus H^{\geq 0}$  and let  $a \in F$  with  $v(a) = \alpha$ . Note that  $a \in P$  since  $v(a) \notin H^{\geq 0}$ . By the definition of  $N$ , there exists  $n \in \mathbb{N}$  such that

$$w(x^n) \geq nw(x) \geq v(a).$$

Therefore

$$x^n \in aO_w \subseteq PO_w \subseteq Q$$

and thus  $x \in Q$ , a contradiction.

Q.E.D.

For a monoid  $M$  and subsets  $N_1, N_2 \subseteq M$ , we denote

$$N_1 + N_2 = \{m_1 + m_2 \mid m_1 \in N_1, m_2 \in N_2\}.$$

**Lemma 4.12.**  $O_w$  satisfies GD over  $O_v$ .

*Proof.* Let  $P_1 \subset P_2 \in \text{Spec}(O_v)$  and let  $Q_2$  be a prime ideal of  $O_w$  such that  $Q_2 \cap O_v = P_2$ . We shall prove that there exists a  $Q_1 \in \text{Spec}(O_w)$  such that  $Q_1 \cap O_v = P_1$  and  $Q_1 \subset Q_2$ . We denote  $S_1 = O_v \setminus P_1$ ,  $S_2 = O_w \setminus Q_2$  and

$$S = \{s_1 s_2 \mid s_1 \in S_1, s_2 \in S_2\}.$$

Note that  $S$  is a multiplicative monoid. We shall prove  $S \cap P_1 O_w = \emptyset$  and then every ideal  $Q_1$  which contains  $P_1 O_w$ , maximal with respect to  $S \cap Q_1 = \emptyset$ , is prime. Note that such  $Q_1$  satisfies the required properties,  $Q_1 \cap O_v = P_1$  and  $Q_1 \subset Q_2$ .

Let  $H_i \leq \Gamma_v$  ( $i = 1, 2$ ) be the isolated subgroups corresponding to  $P_i$  and let  $\text{hull}_{M_w}((H_i)^{\geq 0})$  be their corresponding PIMs in  $(M_w)^{\geq 0}$ .

Let  $x \in P_1 O_w$  and write, by Lemma 3.10,  $x = pa$  where  $p \in P_1$ ,  $a \in O_w$ . Then

$$w(x) = w(pa) \geq w(p) = v(p) \notin H_1.$$

We claim  $w(x) \notin \text{hull}_{M_w}((H_1)^{\geq 0})$ . Indeed, otherwise,  $w(x) \leq \beta$  for some  $\beta \in (H_1)^{\geq 0}$ , a contradiction.

Now, let  $y \in S$ , write  $y = ab$  for  $a \in S_1$ ,  $b \in S_2$ . By Lemma 4.11,

$$w(b) \in \overline{\text{hull}_{M_w}((H_2)^{\geq 0})}.$$

As for  $a$ ,  $a \in S_1$  and thus  $w(a) = v(a) \in H_1$ . In particular  $w(a) \in \text{hull}_{M_w}((H_1)^{\geq 0})$ . Note that  $P_1 \subset P_2$  and thus  $H_2 \subset H_1$  and  $\text{hull}_{M_w}((H_2)^{\geq 0}) \subset \text{hull}_{M_w}((H_1)^{\geq 0})$ . Thus,

$$\overline{\text{hull}_{M_w}((H_2)^{\geq 0})} \subset \text{hull}_{M_w}((H_1)^{\geq 0}).$$

(Indeed, since  $H_2 \subset H_1$ ,  $\text{hull}_{M_w}((H_1)^{\geq 0})$  contains an  $\alpha \in (H_1)^{\geq 0} \setminus (H_2)^{\geq 0}$  and any PIM  $N_2$  lying over  $(H_2)^{\geq 0}$  does not contain  $\alpha$ ). Also note that  $a$  is stable with respect to  $w$ . So we have

$$w(ab) = w(a) + w(b) \in \text{hull}_{M_w}((H_1)^{\geq 0}) + \overline{\text{hull}_{M_w}((H_2)^{\geq 0})} = \text{hull}_{M_w}((H_1)^{\geq 0}).$$

So, we have proved that

$$w(S) \subseteq \text{hull}_{M_w}((H_1)^{\geq 0})$$

and

$$w(P_1 O_w) \subseteq (M_w)^{\geq 0} \setminus \text{hull}_{M_w}((H_1)^{\geq 0}).$$

Therefore  $S \cap P_1 O_w = \emptyset$ .

Q.E.D

Note: it is possible to prove GD in the more general case in which  $E$  is a finite dimensional  $F$ -algebra, but the proof is more complicated and uses the notion of filter quasi-valuations (to be introduced in section 9).



**Corollary 4.13.**  $K\text{-dim}O_w = K\text{-dim}O_v$ .

*Proof.* By Remark 3.5 and Theorem 3.7,  $O_w$  satisfies INC over  $O_v$ ; thus, we get  $K\text{-dim}O_w \leq K\text{-dim}O_v$ . By Lemma 3.12,  $O_w$  satisfies LO over  $O_v$  and by Lemma 4.12,  $O_w$  satisfies GD over  $O_v$ ; thus  $K\text{-dim}O_w \geq K\text{-dim}O_v$ .

Q.E.D.

If  $R$  is a ring and  $P \in \text{Spec}(R)$ , we write  $h_R(P)$  for the height of  $P$  in  $R$ . Note that if  $R$  is a valuation ring and  $K\text{-dim}R < \infty$  then two prime ideals have the same height iff they are equal, since the ideals are linearly ordered by inclusion.

To avoid complicated notation, we shall write  $h_v(P)$  instead of  $h_{O_v}(P)$  for  $P \in \text{Spec}(O_v)$ .

**Lemma 4.14.** *Let  $(F, v)$  be a valued field with valuation ring  $O_v$  and  $K\text{-dim}O_v < \infty$ . Let  $E/F$  be a finite field extension and let  $R$  be a subring of  $E$  lying over  $O_v$ . Assume that  $R$  satisfies GD over  $O_v$ . Let  $Q \in \text{Spec}(R)$  and  $P \in \text{Spec}(O_v)$ ; then*

$$P = Q \cap O_v \text{ iff } h_R(Q) = h_v(P).$$

*Proof.*  $(\Rightarrow)$  By Remark 3.5 and Theorem 3.7,  $R$  satisfies INC over  $O_v$ ; thus, we get  $h_R(Q) \leq h_v(P)$ . By assumption,  $R$  satisfies GD over  $O_v$ ; thus  $h_v(P) \leq h_R(Q)$ .

$(\Leftarrow)$  Write  $Q \cap O_v = P'$ . Then, by  $(\Rightarrow)$ ,  $h_R(Q) = h_v(P')$  and thus  $h_v(P) = h_v(P')$ . Therefore  $P' = P$ .

Q.E.D.

**Lemma 4.15.** *Let  $Q \in \text{Spec}(O_w)$ ,  $P \in \text{Spec}(O_v)$  and assume  $K\text{-dim}O_v < \infty$ ; then*

$$P = Q \cap O_v \text{ iff } h_w(Q) = h_v(P).$$

*Proof.*  $O_w \subseteq E$  lies over  $O_v$ , and by Lemma 4.12  $O_w$  satisfies GD over  $O_v$ . Now use Lemma 4.14.

Q.E.D.

Let  $R$  be a ring;  $R$  is said to satisfy the height formula if for any ideal  $P \in \text{Spec}(R)$ , we have

$$K\text{-dim}R = h_R(P) + K\text{-dim}(R/P).$$

The following lemma is a well known result, seen by matching chains of prime ideals:

**Lemma 4.16.** *If  $C \subseteq R$  are commutative rings such that  $C$  satisfies the height formula and  $R$  satisfies GU, GD and INC over  $C$ , then  $R$  satisfies the height formula.*

In Theorem 5.16 we shall prove that if  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$  (and thus  $M_w$  is actually a group) then  $O_w$  satisfies GU over  $O_v$ .

Since  $O_w$  satisfies INC and GD over  $O_v$ , we deduce by Lemma 4.16 the following theorem,

**Theorem 4.17.** *If  $O_w$  satisfies GU over  $O_v$  then  $O_w$  satisfies the height formula.*

**Lemma 4.18.** *Let  $(F, v)$  be a valued field with valuation ring  $O_v$ . Let  $E$  be a finite field extension and let  $R$  be a subring of  $E$  lying over  $O_v$ . Assume that  $R$  satisfies GD over  $O_v$ . Let  $I$  be an ideal of  $R$  containing  $P \in \text{Spec}(O_v)$  and assume that  $I \subseteq Q$  for all  $Q \in \mathcal{Q}$  where  $\mathcal{Q} = \{Q \in \text{Spec}(R) \mid Q \cap O_v = P\}$ . Then  $\sqrt{I} = \bigcap_{Q \in \mathcal{Q}} Q$ .*

*Proof.*  $\sqrt{I}$  is a radical ideal and thus  $\sqrt{I} = \bigcap_{j \in J} S_j$  where the  $S_j$  are the ideals in  $\text{Spec}(R)$  containing  $I$ . Now, we must have  $S_j \cap O_v \supseteq P$  for every such  $S_j$  since  $I \supseteq P$ . However, assuming some  $S_{j_0}$  satisfies  $S_{j_0} \cap O_v \supset P$ ; we have, by GD,  $Q_{i_0} \subset S_{j_0}$  for some  $Q_{i_0} \in \mathcal{Q}$ . Thus, since  $I \subseteq Q$  for all  $Q \in \mathcal{Q}$ , we can take each  $S_j$  to be in  $\mathcal{Q}$ ; finally, since  $\sqrt{I} \subseteq \bigcap_{Q \in \mathcal{Q}} Q$ , we have  $\sqrt{I} = \bigcap_{Q \in \mathcal{Q}} Q$ .

Q.E.D.

**Proposition 4.19.** *Let  $P \in \text{Spec}(O_v)$ ,  $H$  its corresponding isolated subgroup in  $\Gamma_v$  and  $\text{hull}_{M_w}(H^{\geq 0})$  the corresponding PIM in  $(M_w)^{\geq 0}$ . Let*

$$I = \{x \in O_w \mid w(x) \notin \overline{\text{hull}_{M_w}(H^{\geq 0})}\}$$

*and let  $\mathcal{Q} = \{Q \in \text{Spec}(O_w) \mid Q \cap O_v = P\}$ . Then*

$$\sqrt{I} = \bigcap_{Q \in \mathcal{Q}} Q.$$

*Proof.* First note that  $P \subseteq I$  since  $a \in P$  implies  $v(a) \notin H$  and thus  $w(a) = v(a) \notin \overline{\text{hull}_{M_w}(H^{\geq 0})}$ . Let  $x \in I$ ; then

$$w(x) \notin \overline{\text{hull}_{M_w}(H^{\geq 0})}$$

and thus by Lemma 4.11,  $x \in Q$  for every  $Q \in \mathcal{Q}$ . Now, by Lemma 4.12,  $O_w$  satisfies GD over  $O_v$ . Finally, use Lemma 4.18.

Q.E.D.

We shall use Proposition 4.19 to prove a similar result in section 6 (Proposition 6.5) where we get that the value monoid is a group.

## §5 THE PRIME SPECTRUM AND GU

In this section we study expansions of  $(O_w, K)$  where  $K$  is a maximal ideal of  $O_w$ . We construct a quasi-valuation on a localization of  $O_w$ , which enables us to prove that  $O_w$  satisfies GU over  $O_v$ . We also obtain a bound on the size of the prime spectrum of  $O_w$ .

We start with the following easy lemma:

**Lemma 5.1.** *Let  $(G, +)$  be an abelian group and let  $M \supseteq G$  be a monoid such that  $M$  is torsion over  $G$ ; i.e., for every  $m \in M$  there exists an  $n \in \mathbb{N}$  such that  $nm \in G$ . Then  $M$  is a group.*

*Proof.* Let  $m \in M$  and take  $n \in \mathbb{N}$  such that  $nm = g \in G$ . Thus  $-g \in G \subseteq M$  and  $-m = -g + (n-1)m \in M$ .

Q.E.D.

*Example 5.2.* Take  $F = \mathbb{Q}$ ,  $v = v_p$ ,  $M = \mathbb{R}$  and  $\gamma = \pi$  in Example 2.9; then  $w(E \setminus \{0\}) = \mathbb{Z} \cup \{z - \pi \mid z \in \mathbb{Z}\}$  is not torsion over  $\Gamma_v = \mathbb{Z}$ .

Note that Example 5.2 indicates that even in the case of a finite field extension,  $M_w$  (the monoid generated by  $w(E \setminus \{0\})$ ) is not necessarily torsion over  $\Gamma_v$ . Also note that (since  $w$  in Example 5.2 is group-valued) one can easily take the subgroup  $\Gamma_w$  of  $\mathbb{R}$  generated by  $w(E \setminus \{0\})$  and get a quasi-valuation with value group that is not torsion over  $\Gamma_v$  (note for example that  $\pi \in \Gamma_w \setminus M_w$  in Example 5.2).

We assume throughout sections 5 and 6 that  $E$  is a finite dimensional field extension with quasi-valuation  $w$  extending  $v$  on  $F$  and  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$  (which is the case in many of the examples). Note this implies that  $M_w$  is torsion over  $\Gamma_v$ . Therefore, by Lemma 5.1,  $M_w$  is a group; we denote it as  $\Gamma_w$ .

We note that some of the results presented in sections 5, 6 and 7 stand only in the case that  $M_w$  is a group. Some of the results remain true in the more general case that  $M_w$  is a monoid, but the proofs are much more complicated. Moreover, there are some new interesting results when  $M_w$  is required to be a monoid; we shall present them in sections 8, 9 and 10.

**Lemma 5.3.** *Let  $K$  be a maximal ideal of  $O_w$ . Then  $I_w \subseteq K$ .*

*Proof.* Assume to the contrary that there exists a nonzero element  $x \in O_w$ ,  $x \notin K$ , with  $w(x) > 0$ . Then  $K + \langle x \rangle = O_w$ ; i.e., there exist  $m \in K$  and  $y \in O_w$  such that

$$(1) \quad m + xy = 1.$$

Since  $w(xy) \geq w(x) + w(y) \geq w(x) > 0$ , one has  $w(m) = 0$ . Furthermore,  $m^{-1} \notin O_w$  implies  $w(m^{-1}) < 0$ . Now, multiplying Equation (1) by  $m^{-1}$  we get

$$1 + xym^{-1} = m^{-1}.$$

So,  $w(xym^{-1}) = w(m^{-1}) < 0$  (since  $w(m^{-1}) < 0$  and  $w(1)=0$ ). Then,

$$w(m^{-1}) = w(xym^{-1}) \geq w(x) + w(y) + w(m^{-1}) \geq w(m^{-1}).$$

Therefore, we have equality. Now, cancel  $w(m^{-1})$  from both sides and get  $w(x) + w(y) = 0$ . Thus  $w(x) = 0$ , a contradiction.

Q.E.D.

**Corollary 5.4.**  $[O_w/J_w : O_v/I_v] \leq n$ .

*Proof.* By Lemma 5.3,  $I_w \subseteq K$  for every maximal ideal  $K$  of  $O_w$ . Thus  $I_w \subseteq J_w$ , and we have the natural epimorphism

$$O_w/I_w \twoheadrightarrow O_w/J_w.$$

The result now follows from Corollary 2.6.

Q.E.D.

**Theorem 5.5.** *The quasi-valuation ring  $O_w$  has  $\leq n$  maximal ideals.*

*Proof.*  $O_w/J_w$  is a semisimple ring. Take any set  $\{K_i\}_{i=1}^t$  of maximal ideals of  $O_w$ . Then  $J_w \subseteq \cap_{i=1}^t K_i$ ; so we have an epimorphism

$$O_w/J_w \twoheadrightarrow \bigoplus_{i=1}^t O_w/K_i.$$

Thus by Corollary 5.4,

$$n \geq [O_w/J_w : O_v/I_v] \geq \sum_{i=1}^t [O_w/K_i : O_v/I_v] \geq \sum_{i=1}^t 1 = t,$$

proving  $t \leq n$ .

Q.E.D.

Note that if  $O_w$  is a valuation ring then  $O_w$  has one maximal ideal.

The proofs of the next two results are standard.

**Lemma 5.6.** *Let  $E/F$  be a field extension,  $O_v$  a valuation ring of  $F$  with valuation ideal  $I_v$  and  $O_u$  a valuation ring of  $E$  containing  $O_v$  with valuation ideal  $I_u$  containing  $I_v$ . Then  $O_u \cap F = O_v$ .*

*Proof.* Assume to the contrary, that there exists an  $a \in (O_u \cap F) \setminus O_v$ . Then

$$a^{-1} \in I_v \subseteq I_u,$$

so  $1 = aa^{-1} \in I_u$ , a contradiction.

Q.E.D.

**Proposition 5.7.** *Suppose  $E/F$  is a finite dimensional field extension and let  $R \subseteq E$  be a ring such that  $R \cap F = O_v$ . Let  $K$  be a maximal ideal of  $R$  such that  $K \supseteq I_v$ ; then there exists a valuation  $u$  of  $E$  that extends  $v$  such that  $O_u \supseteq R$  and  $I_u \supseteq K$ .*

*Proof.* We look at the pair  $(R, K)$  and we take the collection of all pairs  $(R_\alpha, I_\alpha)$  where  $R \subseteq R_\alpha \subseteq E$  and  $K \subseteq I_\alpha \triangleleft R_\alpha$ . We order these pairs with the partial order of containment, i.e.,

$$(R_\alpha, I_\alpha) \leq (R_\beta, I_\beta) \text{ iff } R_\alpha \subseteq R_\beta \text{ and } I_\alpha \subseteq I_\beta.$$

Zorn's Lemma is applicable to yield a maximal pair  $(O_u, I_u)$ . We claim that  $O_u$  is a valuation ring. Indeed, let  $x \in E$  and assume to the contrary that  $x, x^{-1} \notin O_u$ ; then, by [Kap, p. 35, Th. 55],  $I_u$  is a proper ideal of  $O_u[x]$  or  $O_u[x^{-1}]$ . Either way, we have contradicted the maximality of  $(O_u, I_u)$ . Now, by Lemma 5.6,  $O_u \cap F = O_v$ ; i.e., the valuation  $u$  which corresponds to  $O_u$  extends  $v$ .

Q.E.D.

*Definition 5.8.* We call such a maximal pair  $(O_u, I_u)$  obtained in Proposition 5.7 an *expansion* of  $(R, K)$  to  $E$ . In other words,  $(O_u, I_u)$  is an expansion of  $(R, K)$  to  $E$  if  $R \subseteq O_u \subseteq E$ ,  $K \subseteq I_u$  ( $I_u$  is a proper ideal of  $O_u$ ) and for any pair  $(R', K')$  satisfying  $O_u \subseteq R' \subseteq E$  and  $I_u \subseteq K'$  ( $K'$  is a proper ideal of  $R'$ ), we have  $O_u = R'$  and  $I_u = K'$ . We suppress  $I_u$  when it is not relevant.

**Corollary 5.9.** *There exists a valuation  $u$  on the field  $E$  that extends  $v$ , such that*

$$O_u \supseteq O_w.$$

*Proof.* Take  $R = O_w$  in Proposition 5.7; take any maximal ideal  $K$  of  $O_w$ , and note that  $K \supseteq I_w$  by Lemma 5.3.

Q.E.D.

Note that one can expand the pair  $(O_w, K_i)$  for every maximal ideal  $K_i$  of  $O_w$ . Also note that if  $K_i \neq K_j$  then  $(O_{u_i}, I_{u_i}) \neq (O_{u_j}, I_{u_j})$  where  $(O_{u_i}, I_{u_i})$  is the expansion of  $(O_w, K_i)$  and  $(O_{u_j}, I_{u_j})$  is the expansion of  $(O_w, K_j)$ . There is a well known theorem from valuation theory (cf. [End, p. 97, Th. 13.7]) which says: let

$$\mathcal{U} = \{O_u \subseteq E \mid O_u \text{ is a valuation ring of } E \text{ and } O_u \cap F = O_v\};$$

then  $|\mathcal{U}| \leq [E : F]_{\text{sep}}$ .

We denote  $\mathcal{K} = \{\text{maximal ideals of } O_w\}$  and

$$\mathcal{U}_w = \{O_u \subseteq E \mid O_u \text{ is an expansion of } (O_w, K) \text{ for some } K \in \mathcal{K}\}.$$

We have:

*Remark 5.10.*  $\mathcal{U}_w \subseteq \mathcal{U}$ .

*Proof.* Let  $O_u \in \mathcal{U}_w$ ; then  $O_u$  is a valuation ring of  $E$  with valuation ideal  $I_u$ . Now, by the definition of  $\mathcal{U}_w$ ,  $O_u \supseteq O_w \supseteq O_v$  and  $I_u \supseteq K$  for some maximal ideal of  $O_w$ . By Lemma 5.3, every maximal ideal  $K \triangleleft O_w$  contains  $I_w$  which contains  $I_v$ ; thus  $I_u \supseteq I_v$ . Therefore, by Lemma 5.6,  $O_u \in \mathcal{U}$ .

**Corollary 5.11.**  $|\mathcal{K}| \leq |\mathcal{U}_w| \leq |\mathcal{U}| \leq [E : F]_{\text{sep}}$ .

For a ring  $R \subseteq E$  we denote by  $I_E(R)$  the integral closure of  $R$  in  $E$ .

The following remark is known from valuation theory. (See, for example, [End, p.69, Th. 10.8]):

*Remark 5.12.* Let  $R$  be a subring of  $E$ , then  $I_E(R) = \bigcap \{O_u \mid O_u \text{ is a valuation ring of } E \text{ containing } R\} = \bigcap \{O_u \mid O_u \text{ is a valuation ring of } E \text{ containing } R, \text{ and } I_u \cap R \text{ is a maximal ideal of } R\}$ . In particular,  $I_E(O_w) = \bigcap \{O_u \mid O_u \text{ is a valuation ring of } E \text{ containing } O_w, \text{ and } I_u \cap O_w \text{ is a maximal ideal of } O_w\}$ .

*Remark 5.13.*  $\mathcal{U}_w = \{O_u \mid O_u \text{ is a valuation ring of } E \text{ containing } O_w, \text{ and } I_u \cap O_w \text{ is a maximal ideal of } O_w\}$ . Indeed,  $\subseteq$  is obvious. Conversely, if  $O_u \supseteq O_w$  is a valuation ring of  $E$  and  $I_u \cap O_w$  is a maximal ideal of  $O_w$ , then  $(O_u, I_u)$  is an element in the collection described in Proposition 5.7. Assuming  $(O_u, I_u) \leq (O_{u'}, I_{u'})$  for some ring  $O_{u'} \subseteq E$  and a maximal ideal  $I_{u'} \triangleleft O_{u'}$ , we conclude that  $O_{u'}$  is a valuation ring of  $E$ . Thus,  $O_u \subset O_{u'}$  implies  $I_{u'} \subset I_u$ ; so we must have  $O_u = O_{u'}$  and  $I_u = I_{u'}$ . I.e.,  $O_u \in \mathcal{U}_w$ . We conclude, by Remark 5.12, that

$$I_E(O_w) = \bigcap_{O_u \in \mathcal{U}_w} O_u.$$

**Lemma 5.14.** Let  $O_u \in \mathcal{U}_w$  and let  $I_u$  be its maximal ideal; then  $I_u \cap I_E(O_w)$  is a maximal ideal of  $I_E(O_w)$ .

*Proof.*  $I_u \cap I_E(O_w)$  is a prime ideal of  $I_E(O_w)$ ; thus  $(I_u \cap I_E(O_w)) \cap O_w$  is a prime ideal of  $O_w$ . However,  $I_u \cap O_w$  is a maximal ideal in  $O_w$ . (Indeed,  $I_u$  contains a maximal ideal of  $O_w$ ). Now, by [Row, p.187, Corollary 6.33] (which says for  $R$  integral over  $C$  and  $Q \in \text{Spec}(R)$  lying over  $P \in \text{Spec}(C)$ , that  $P$  is maximal in  $C$  iff  $Q$  is maximal in  $R$ ),  $I_u \cap I_E(O_w)$  is a maximal ideal of  $I_E(O_w)$ .

Q.E.D.

Let  $H$  be an isolated subgroup of a totally ordered abelian group  $\Gamma$ . Recall that the quotient group  $\overline{\Gamma} = \Gamma/H$  is totally ordered, by setting  $\overline{\gamma} \geq \overline{h}$  iff  $\gamma \geq h$  for some  $h \in H$ . There is a one to one correspondence between the set of all prime ideals of a valuation ring and the set  $G(\Gamma)$  of all isolated subgroups of  $\Gamma$ . The rank of  $\Gamma$  is the order type of  $G(\Gamma) \setminus \{\Gamma\}$  (see [End, p. 47]).

We note that for totally ordered abelian groups  $\Gamma \subseteq \Delta$  and an isolated subgroup  $H$  of  $\Gamma$ , the convex hull of  $H$  in  $\Delta$ ,  $\text{hull}_\Delta(H)$ , is an isolated subgroup of  $\Delta$  (we call it the corresponding isolated subgroup of  $H$  in  $\Delta$ ).

Also recall that if  $\Gamma$  and  $\Delta$  are totally ordered abelian groups such that  $\Gamma \subseteq \Delta$  and  $\Delta/\Gamma$  is a torsion group, then  $\text{rank}\Gamma = \text{rank}\Delta$ ; i.e., there is a one to one correspondence between the set of isolated subgroups of  $\Gamma$  and the set of isolated subgroups of  $\Delta$ . It is easy to see (when  $\Delta/\Gamma$  is a torsion group) that

$$\text{hull}_\Delta(H) = \{\delta \in \Delta \mid n\delta \in H \text{ for some } n \in \mathbb{N}\}.$$

See [End, Chapter 13] or [Bo, Section 4] for further discussion.

**Theorem 5.15.** *Let  $P$  be a prime ideal of  $O_v$ ,  $H \leq \Gamma_v$  its corresponding isolated subgroup and  $\text{hull}_{\Gamma_w}(H) \leq \Gamma_w$  the corresponding isolated subgroup. Let  $f : \Gamma_w \rightarrow \Gamma_w/\text{hull}_{\Gamma_w}(H)$  be the natural epimorphism and let*

$$\tilde{w} : E \rightarrow (\Gamma_w/\text{hull}_{\Gamma_w}(H)) \cup \{\infty\}$$

*be a map such that  $\tilde{w}(0) = \infty$  and  $\tilde{w}(x) = f(w(x)) = w(x) + \text{hull}_{\Gamma_w}(H)$  for  $x \neq 0$ . Then  $\tilde{w}$  is a quasi-valuation on  $E$  satisfying*

- (1)  $O_w^\sim \supseteq O_w$ .
- (2)  $\tilde{w}$  extends  $\tilde{v}$ , the corresponding valuation of  $(O_v)_P$ , and  $O_w^\sim$  lies over  $(O_v)_P$ , i.e.,  $O_w^\sim \cap F = (O_v)_P$ .
- (3)  $O_w^\sim = O_w S^{-1}$  where  $S = O_v \setminus P$ .

*Proof.* Note that  $f$  is an epimorphism of ordered groups, so  $\alpha \leq \beta \in \Gamma_w$  implies  $f(\alpha) \leq f(\beta)$ . Now, let  $x, y$  be two nonzero elements of  $E$ . We obviously have  $w(xy) \geq w(x) + w(y)$  and thus

$$\begin{aligned} \tilde{w}(xy) &= f(w(xy)) \geq f(w(x) + w(y)) \\ &= f(w(x)) + f(w(y)) = \tilde{w}(x) + \tilde{w}(y). \end{aligned}$$

Next, assume  $\tilde{w}(x) \leq \tilde{w}(y)$ ; we have

$$\tilde{w}(x + y) = f(w(x + y)) \geq f(\min\{w(x), w(y)\}).$$

Now, if  $\tilde{w}(x) = \tilde{w}(y)$  then  $f(w(x)) = f(w(y))$  and thus

$$f(\min\{w(x), w(y)\}) = f(w(x)) = \tilde{w}(x).$$

If  $\tilde{w}(x) < \tilde{w}(y)$  then  $w(x) < w(y)$  and again

$$f(\min\{w(x), w(y)\}) = f(w(x)) = \tilde{w}(x).$$

So, we have

$$\tilde{w}(x+y) \geq \min\{\tilde{w}(x), \tilde{w}(y)\}.$$

Note that if  $x \in O_w$  then  $w(x) \geq 0$  and thus  $f(w(x)) \geq \bar{0}$ , so  $x \in O_w^\sim$  and (1) is proved.

To prove (2) we denote by  $g : \Gamma_v \rightarrow \Gamma_v/H$  the natural epimorphism and note that  $\forall x \neq 0, \tilde{v}(x) = g(v(x)) = v(x) + H$ . Also note that one can view  $\Gamma_v/H$  inside  $\Gamma_w/\text{hull}_{\Gamma_w}(H)$  via the natural monomorphism. In this sense,  $\tilde{w}$  is an extension of  $\tilde{v}$ . Now, the fact that  $O_w^\sim \cap F = (O_v)_P$  follows immediately.

We now prove (3). Let  $rs^{-1} \in O_w S^{-1}$  then  $w(s) = v(s) \in H$ . Write  $w(s) = h$ . We have

$$w(rs^{-1}) = w(r) - w(s) \geq -h \in H.$$

Hence  $\tilde{w}(rs^{-1}) \geq \bar{0}$  and  $rs^{-1} \in O_w^\sim$ . On the other hand, let  $x \in O_w^\sim$ ; then  $\tilde{w}(x) \geq \bar{0}$  i.e.,  $w(x) \geq j$  for some  $j \in \text{hull}_{\Gamma_w}(H)$ . Note that if  $x \in O_w$  it is obvious that  $x \in O_w S^{-1}$ . So, we may assume  $x \in O_w^\sim \setminus O_w$ . We write  $w(x) = -h'$  for some  $h' \in (\text{hull}_{\Gamma_w}(H))^{\geq 0}$ . (Indeed if  $w(x) < -h'$  for all  $h' \in (\text{hull}_{\Gamma_w}(H))^{\geq 0}$  then  $x \notin O_w^\sim$ ). We take  $t \in \mathbb{N}$  such that  $th' \in H$  and pick  $s \in S$  such that  $v(s) = th'$ , then

$$w(xs) = w(x) + w(s) = (t-1)h' \geq 0.$$

Thus  $xs \in O_w$  and  $x = (xs)s^{-1}$  for  $xs \in O_w$  and  $s \in O_v \setminus P$ .

Q.E.D.

We say that  $R$  satisfies GU over  $C$  if for any  $P_0 \subseteq P_1$  in  $\text{Spec}(C)$  and every  $Q_0 \in \text{Spec}(R)$  lying over  $P_0$  there is  $Q_0 \subseteq Q_1$  in  $\text{Spec}(R)$  lying over  $P_1$ ; cf. [Row, p. 185]

**Theorem 5.16.**  $O_w$  satisfies GU over  $O_v$ .

*Proof.* Let  $P_0 \subseteq P_1 \in \text{Spec}(O_v)$  and assume  $Q_0 \in \text{Spec}(O_w)$  such that  $Q_0 \cap O_v = P_0$ . We prove that there exists a  $Q_0 \subseteq Q \in \text{Spec}(O_w)$  such that  $Q \cap O_v = P_1$ . By [Row, p.186, Lemma 6.30] there exists an ideal  $Q \supseteq Q_0$ , maximal with respect to  $Q \cap O_v \subseteq P_1$  and any such  $Q$  is in  $\text{Spec}(O_w)$ . We shall prove that  $Q \cap O_v = P_1$ . Denote  $S_1 = O_v \setminus P_1$ ; then, by Theorem 5.15,  $O_w S_1^{-1}$  has a quasi-valuation  $\tilde{w}$  with value group such that  $\tilde{w}$  extends the valuation corresponding to  $O_v S_1^{-1}$ . Now, from the maximality of  $Q$  and the injective order preserving mapping between ideals of  $O_w S_1^{-1}$  and ideals of  $O_w$  disjoint from  $S_1$ , we deduce that  $Q S_1^{-1}$  is a maximal ideal of  $O_w S_1^{-1}$ . However, by Lemma 5.3, every maximal ideal of  $O_w S_1^{-1}$  must contain  $P_1 S_1^{-1}$ . Therefore  $Q \cap O_v = P_1$ .

Q.E.D.

We are now able to deduce Corollary 4.13 (in the special case where the value monoid of the quasi-valuation is a group) without using the going down property.

**Corollary 5.17.**  $K\text{-dim} O_w = K\text{-dim} O_v$ .

*Proof.* By Remark 3.5 and Theorem 3.7,  $O_w$  satisfies INC over  $O_v$ ; by Theorem 5.16,  $O_w$  satisfies GU over  $O_v$ . Thus by [Row, p.185, Remark 6.26],  $K\text{-dim} O_w = K\text{-dim} O_v$ .

Q.E.D.

It is known from valuation theory (see [End, p.102, Th. 13.14]) that for every valuation ring  $O_u \subseteq E$  such that  $O_u \cap F = O_v$ , one has

$$\text{K-dim} O_v = \text{K-dim} O_u.$$

Also, it is well known that if  $R$  is integral over  $C$ , then  $\text{K-dim} R = \text{K-dim} C$ . So, we have the following Theorem:

**Theorem 5.18.**  $K\text{-dim} O_v = K\text{-dim} O_w = K\text{-dim} I_E(O_w) = K\text{-dim} O_u$ .

At this point, in preparation for computing Krull dimension in the next theorem, we pause to build another quasi-valuation  $w'$  which arises naturally from  $w$ . Let  $O_u$  be a valuation ring of  $E$  such that  $O_u \cap F = O_v$ . We first note that since  $E$  is algebraic over  $F$ ,  $\Gamma_u$  is torsion over  $\Gamma_v$  (see [End, p.99-100, Thm.13.9 and 13.11]). We recall our assumption that  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$ . Therefore  $\Gamma_u$  and  $\Gamma_w$  can be embedded in  $\Gamma_{\text{div}}$ , where  $\Gamma_{\text{div}}$  is the divisible hull of  $\Gamma_v$ . Note that  $\Gamma_{\text{div}}/\Gamma_v$  is a torsion group, while on the other hand, for any totally ordered abelian group  $\Delta$  containing  $\Gamma_v$  and such that  $\Delta/\Gamma_v$  is a torsion group, there is an embedding of  $\Delta$  in  $\Gamma_{\text{div}}$ . For more details on  $\Gamma_{\text{div}}$  see [End, p.100-102].

Now, we define  $w' : E \rightarrow \Gamma_{\text{div}} \cup \{\infty\}$  by

$$w'(x) = \min\{u(x) \mid u \text{ is a valuation whose valuation ring is in } \mathcal{U}_w\}.$$

Note that  $|\mathcal{U}_w| < \infty$ , so  $w'$  is well defined. Also note that  $w'$  is an exponential quasi-valuation on  $E$  that extends  $v$  and  $O_{w'}$  is the intersection of all  $O_u \in \mathcal{U}_w$ . Thus, by Remark 5.13,  $O_{w'} = I_E(O_w)$ .

**Theorem 5.19.** *Let  $O_v \subseteq R \subseteq E$  be a ring whose Jacobson radical satisfies  $J \supseteq I_v$ . Then,  $K\text{-dim} R = K\text{-dim} O_v$ , and  $R$  has a finite number of maximal ideals.*

*Proof.* Let  $\mathcal{U}_R = \{O_u \mid O_u \text{ is a valuation ring of } E \text{ containing } R \text{ and } I_u \cap R \text{ is a maximal ideal of } R\}$ . By Remark 5.12 we have

$$I_E(R) = \bigcap_{O_u \in \mathcal{U}_R} O_u.$$

Now, Let  $O_u \in \mathcal{U}_R$  and let  $I_u$  denote its maximal ideal; since  $I_u \cap R$  is a maximal ideal of  $R$ , we get  $I_u \cap O_v = I_v$  (since  $I_u \supseteq J \supseteq I_v$ ) and thus by Lemma 5.6,  $O_u \cap F = O_v$ . Therefore,  $\mathcal{U}_R \subseteq \mathcal{U}$ .

As above, we define  $w'' : E \rightarrow \Gamma_{\text{div}} \cup \{\infty\}$  by

$$w''(x) = \min\{u(x) \mid u \text{ is a valuation on } E \text{ whose valuation ring is in } \mathcal{U}_R\}.$$

$w''$  is well defined since  $|\mathcal{U}_R| < \infty$ . Also,  $w''$  is an exponential quasi-valuation on  $E$  that extends  $v$  and since  $O_{w''}$  is the intersection of all  $O_u \in \mathcal{U}_R$ , we have  $O_{w''} = I_E(R)$ .

By Corollary 4.13 (or Corollary 5.17) we get,

$$\text{K-dim} I_E(R) = \text{K-dim} O_v$$

and since  $I_E(R)$  is integral over  $R$ , we get

$$\text{K-dim} I_E(R) = \text{K-dim} R.$$



A similar proof to the proof of Remark 5.13 shows that

$$\mathcal{U}_R = \{O_u \mid O_u \text{ is an expansion of } (R, K) \text{ for some maximal ideal } K \text{ of } R\}.$$

Now assume that  $R$  has an infinite number of maximal ideals; then we get an infinite number of  $O_u$ 's all of which lie over  $O_v$ , a contradiction.

Q.E.D.

We note that since  $I_E(O_w)$  is integral over  $O_w$ , there is a surjective mapping  $\text{Spec}(I_E(O_w)) \rightarrow \text{Spec}(O_w)$  given by  $K \rightarrow K \cap O_w$  sending  $\{\text{maximal ideals of } I_E(O_w)\} \rightarrow \{\text{maximal ideals of } O_w\}$ . See [Row, p. 184-189] for further discussion.

**Lemma 5.20.**  $\mathcal{U}_w = \mathcal{U}'_w$  where  $\mathcal{U}'_w = \{O_u \mid O_u \text{ is an expansion of } (I_E(O_w), K) \text{ for some maximal ideal } K \text{ of } I_E(O_w)\}$ .

*Proof.* Let  $O_u \in \mathcal{U}'_w$ . Then  $O_u \supseteq I_E(O_w) \supseteq O_w$ , and  $I_u \supseteq K' \supseteq K$  for maximal ideals  $K'$  and  $K$  in  $I_E(O_w)$  and  $O_w$  respectively. So,  $O_u \in \mathcal{U}_w$ . Conversely, if  $O_u \in \mathcal{U}_w$ , then  $I_u \cap I_E(O_w)$  is a maximal ideal of  $I_E(O_w)$  by Lemma 5.14, so  $(I_E(O_w), I_u \cap I_E(O_w))$  has an expansion  $(O_u, I_u)$ .

Q.E.D.

Let  $P \triangleleft O_v$  be a prime ideal and let  $S = O_v \setminus P$ ; then  $S^{-1}O_v$  is a valuation ring of  $F$  with valuation ideal  $S^{-1}PO_v$ . Note that  $S^{-1}O_w \supseteq S^{-1}O_v$  and every maximal ideal of  $S^{-1}O_w$  contains  $S^{-1}PO_v$ ; thus by Theorem 5.19,  $\text{K-dim } S^{-1}O_v = \text{K-dim } S^{-1}O_w$ . (Note that one can also deduce this equation by Theorem 5.15 and Corollary 4.13 (or Corollary 5.17)). Let  $Q \triangleleft O_w$  be a prime ideal lying over  $P$ . We expand  $(S^{-1}O_w, S^{-1}QO_w)$  for every prime ideal  $Q$  lying over  $P$ , to get valuation rings lying over  $S^{-1}O_v$ . Note that if  $Q \neq Q'$  lie over  $P$  then we get different valuation rings. Indeed, assume to the contrary that we get  $(U, I)$  from both of the expansions; then  $I \cap S^{-1}O_w$ , which is a proper ideal of  $S^{-1}O_w$ , contains the maximal ideals  $S^{-1}QO_w$  and  $S^{-1}Q'O_w$ , a contradiction. So, we have

$$\begin{aligned} & (*) \mid \{Q \mid Q \text{ is a prime ideal of } O_w \text{ lying over } P\} \mid \\ & \leq \mid \{U \mid U \text{ is an expansion of } (S^{-1}O_w, K) \text{ for some maximal ideal } K \text{ of } S^{-1}O_w\} \mid \\ & \leq [E : F]_{\text{sep}}. \end{aligned}$$

The last inequality is since, as before, every such  $U$  lies over the valuation ring  $S^{-1}O_v$ . So, we have the following theorem:

**Theorem 5.21.**

$$\text{K-dim } O_v \leq \mid \text{Spec}(O_w) \mid \leq \mid \text{Spec}(I_E(O_w)) \mid \leq [E : F]_{\text{sep}} \cdot \text{K-dim } O_v.$$

*Proof.* The first inequality is due to Lemma 3.12. The second inequality is due to the fact that  $I_E(O_w)$  is integral over  $O_w$ . For the last inequality, note that  $I_E(O_w) = O_{w'}$  where  $w'$  is the quasi-valuation extending  $v$  defined by  $w'(x) = \min\{u(x) \mid u \text{ is a valuation whose valuation ring is in } \mathcal{U}_w\}$  for all  $x \in E$ . ( $w'$  was defined before Theorem 5.19). For every  $P \in \text{Spec}(O_v)$  denote

$$\mathcal{Q}_P = \{Q \in \text{Spec}(O_{w'}) \mid Q \cap O_v = P\};$$

thus  $\text{Spec}(O_{w'}) = \bigcup_{P \in \text{Spec}(O_v)} \mathcal{Q}_P$ . (In fact  $\{\mathcal{Q}_P\}_{P \in \text{Spec}(O_v)}$  is a partition of  $\text{Spec}(O_{w'})$ .) Finally, by (\*) we get

$$\mid \bigcup_{P \in \text{Spec}(O_v)} \mathcal{Q}_P \mid \leq \text{K-dim } O_v \cdot [E : F]_{\text{sep}}.$$

Q.E.D.

## §6 A BOUND ON THE QUASI-VALUATION AND THE HEIGHT FORMULA

In this section we continue to assume that  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$  (and thus the value monoid of the quasi-valuation is a group). We show that any such quasi-valuation on  $E$  extending  $v$  is dominated by any valuation  $u$  on  $E$  extending  $v$  satisfying  $O_u \supseteq O_w$ . We also prove that the quasi-valuation ring satisfies the height formula.

**Lemma 6.1.** *Let  $w$  be a quasi-valuation on  $E$  extending  $v$  and let  $u$  be a valuation on  $E$  extending  $v$  such that  $O_u \supseteq O_w$ ; then  $I_u \supseteq I_w$ .*

*Proof.*  $K = I_u \cap O_w$  is a prime ideal of  $O_w$ . Assuming  $K$  is not maximal in  $O_w$ , one has  $K \cap O_v$  not maximal in  $O_v$  (by INC). This contradicts the fact that  $I_u \cap O_v = I_v$ . So  $I_u \cap O_w$  is a maximal ideal in  $O_w$  and thus (Lemma 5.3) contains  $I_w$ .

Q.E.D.

The following lemma is valid without the assumption that  $\Gamma_w$  and  $\Gamma_{w'}$  are torsion over  $\Gamma_v$ .

**Lemma 6.2.** *Let  $w$  and  $w'$  be quasi-valuations on  $E$  extending  $v$  such that  $\Gamma_w$  and  $\Gamma_{w'}$  embed in a group  $G$  and assume that  $w$  is an exponential quasi-valuation. If there exists  $x \in E$  such that  $w(x) < w'(x)$ , then there exists  $y \in E$  such that  $0 = w(y) < w'(y)$ .*

*Proof.* Write  $\sum_{i=0}^n \alpha_i x^i = 0$  for  $\alpha_i \in F$ . Since the sum is zero we must have  $k, l \in \mathbb{N}$  such that

$$w(\alpha_k x^k) = w(\alpha_l x^l).$$

Assuming  $k < l$  we have, using the fact that  $w$  is exponential,  $w(x^{l-k} \alpha_l \alpha_k^{-1}) = 0$ . Now, since  $w(x) < w'(x)$ , we have

$$w'(x^{l-k}) \geq (l-k)w'(x) > (l-k)w(x) = w(x^{l-k}).$$

Therefore  $w'(x^{l-k} \alpha_l \alpha_k^{-1}) > w(x^{l-k} \alpha_l \alpha_k^{-1}) = 0$ . So take  $y = x^{l-k} \alpha_l \alpha_k^{-1}$ .

Q.E.D.

We now show that the values of the quasi-valuation are bounded by the values of a valuation.

**Theorem 6.3.** *Let  $w$  be a quasi-valuation on the field  $E$  extending a valuation  $v$ . Then there exists a valuation  $u$  extending  $v$  such that  $u$  dominates  $w$  i.e.,  $\forall x \in E$ ,  $w(x) \leq u(x)$ . Moreover, for every valuation  $u$  on  $E$  extending  $v$  satisfying  $O_u \supseteq O_w$ ,  $u$  dominates  $w$ .*

*Proof.* First note that  $\Gamma_u$  and  $\Gamma_w$  embed in  $\Gamma_{\text{div}}$ , so we refer to the ordering in  $\Gamma_{\text{div}}$ . By Corollary 5.9, there exists a valuation  $u$  that extends  $v$ , such that  $O_u \supseteq O_w$ . We now prove that every such  $u$  dominates  $w$ .

Let  $x \in E$  and assume to the contrary that  $u(x) < w(x)$ . Note that  $u$  is a valuation and thus in particular an exponential quasi-valuation. Therefore, by Lemma 6.2 there exists  $y \in E$  such that  $0 = u(y) < w(y)$ , i.e.,  $y \in I_w \setminus I_u$ , in contradiction to Lemma 6.1.

Q.E.D.

The following theorem is a special case of Lemma 4.16 (by taking  $R = O_w$  and  $C = O_v$ ) in view of the previous results. We shall prove it here for the reader's convenience.

**Theorem 6.4.**  $O_w$  satisfies the height formula.

*Proof.* First note that  $O_v$  satisfies the height formula because  $\text{Spec}(O_v)$  is linearly ordered. Also note that if  $\text{K-dim} O_v = \infty$  then by Corollary 4.13 (or Corollary 5.17),  $\text{K-dim} O_w = \infty$ . In this case, if  $Q \in \text{Spec}(O_w)$  has infinite height, then obviously  $\text{K-dim} O_w = h_w(Q) + \text{K-dim}(O_w/Q)$ ; if  $Q \in \text{Spec}(O_w)$  has finite height  $t$  then  $P = Q \cap O_v \in \text{Spec}(O_v)$  has finite height  $t$  by INC and GD and thus by GU,  $\text{K-dim}(O_w/Q) = \infty$ . So, we may assume that  $\text{K-dim} O_v = m$  and by Corollary 4.13 (or Corollary 5.17), we have  $\text{K-dim} O_w = m$ . Now, let  $Q$  be a prime ideal of  $O_w$  and  $P = Q \cap O_v \in \text{Spec}(O_v)$ ; then, by Lemma 4.15,  $h_w(Q) = h_v(P)$ . Write  $h_w(Q) = t$ ,  $Q = Q_t$ ,  $P = P_t$ . We have a chain

$$P_t \subset P_{t+1} \subset \dots \subset P_m = I_v$$

of size  $m - t + 1$  and by GU, we have a chain

$$Q_t \subset Q_{t+1} \subset \dots \subset Q_m.$$

Assume to the contrary that there exists a longer chain. Let  $Q'_t \subset Q'_{t+1} \subset \dots \subset Q'_l$  be a chain of longer size. We may assume that  $Q'_t = Q_t$ . Therefore, by INC, there exists a chain

$$P'_t \subset P'_{t+1} \subset \dots \subset P'_l$$

where  $P'_t = P_t$  and thus a chain

$$P'_0 \subset P'_1 \subset \dots \subset P'_l \subseteq I_v$$

of size  $> m + 1$ , a contradiction.

Q.E.D.

**Proposition 6.5.** Let  $P \in \text{Spec}(O_v)$ ,  $H$  its corresponding isolated subgroup in  $\Gamma_v$ , and  $\text{hull}_{\Gamma_w}(H)$  the corresponding isolated subgroup in  $\Gamma_w$ . Denote

$$I = \{x \in O_w \mid w(x) \notin \text{hull}_{\Gamma_w}(H)\}$$

and let  $\mathcal{Q} = \{Q \in \text{Spec}(O_w) \mid Q \cap O_v = P\}$ . Then

$$\sqrt{I} = \bigcap_{Q \in \mathcal{Q}} Q.$$

*Proof.* Under the hypothesis that  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$  (and thus the value monoid of the quasi-valuation is a group),  $\text{hull}_{\Gamma_w}(H^{\geq 0})$  is the unique PIM lying over  $H^{\geq 0}$  and thus equals  $\overline{\text{hull}_{\Gamma_w}(H^{\geq 0})}$ . Now, use Proposition 4.19.

Q.E.D.

**Corollary 6.6.**  $\sqrt{I_w} = J_w$ .

*Proof.* First note that by Lemma 5.3 and INC (Remark 3.5 and Theorem 3.7), the set of prime ideals lying over  $I_v$  is exactly the set of maximal ideals of  $O_w$ . Thus, the corollary is a special case of the previous proposition, seen by taking  $P = I_v$ .

Q.E.D.

## §7 EXPONENTIAL QUASI-VALUATIONS

In this section we study exponential quasi-valuations extending a valuation in a finite dimensional field extension. We show that the associated rings are integrally closed; we also show that if two exponential quasi-valuations are not equal then their rings cannot be equal. Finally, we prove that exponential quasi-valuations have a unique form and we obtain a bound on the number of these quasi-valuations.

*Definition 7.1.* An additive monoid  $M$  is called *weakly cancellative* if for any  $a, b \in M$ ,  $a + b = a$  implies  $b = 0$ .

Note that, by Lemma 5.1, every torsion monoid over a group is a group and thus in particular weakly cancellative.

In this section we do not assume that  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$ . Instead, we assume the weaker hypothesis that  $M_w$  (the value monoid) is weakly cancellative.

Recall from Definition 1.7 that a quasi-valuation  $w$  is called exponential if for every  $x \in E$ ,

$$w(x^n) = nw(x), \quad \forall n \in \mathbb{N}.$$

**Lemma 7.2.** *If  $w$  is an exponential quasi-valuation then for each nonzero  $b \in E$  there exists  $t = t(b)$ ,  $1 \leq t \leq n$ , such that  $w(b^t) \in \Gamma_v$ , and*

$$\{w(b^i) + \Gamma_v \mid i \in \mathbb{N}\} = \{w(b^i) + \Gamma_v \mid 0 \leq i < t\}.$$

*Proof.* By Lemma 2.1, the set  $\{1, b, \dots, b^n\}$  satisfies

$$w(b^j) + \Gamma_v = w(b^i) + \Gamma_v$$

for some  $0 \leq i < j \leq n$ . Therefore  $jw(b) + \Gamma_v = iw(b) + \Gamma_v$  and we get  $jw(b) + \alpha = iw(b)$  for some  $\alpha \in \Gamma_v$ . Now, add  $(j - i)w(b)$  to both sides and get

$$jw(b) + \alpha + (j - i)w(b) = jw(b).$$

Thus (since  $M_w$  is weakly cancellative) we get  $\alpha + (j - i)w(b) = 0$ , i.e.,  $w(b^{j-i}) = (j - i)w(b) \in \Gamma_v$ . As for the second part, let  $t \leq k \in \mathbb{N}$  and write  $k = qt + r$  for  $0 \leq r < t$ . We have:

$$w(b^k) = w(b^{qt+r}) = w(b^{qt}) + w(b^r) \in w(b^r) + \Gamma_v.$$

Q.E.D.

**Lemma 7.3.** *If  $w$  is an exponential quasi-valuation and  $M_w$  is weakly cancellative, then  $[M_w : \Gamma_v]$  is finite, and consequently  $M_w$  is a group.*

*Proof.* First note that every element of  $M_w$  is of the form  $\sum_{i=1}^k w(x_i)$  for  $x_i \in E$ . By Lemma 2.1, there exists a set  $\{b_1, \dots, b_m\}$  of  $E$ , where  $m \leq n$ , such that  $\{w(b_i) + \Gamma_v \mid 1 \leq i \leq m\}$  comprises all the cosets of  $\Gamma_v$  in  $w(E \setminus \{0\})$ . Thus, one can replace every coset  $\sum_{i=1}^k w(x_i) + \Gamma_v$  by  $\sum_{i=1}^m n_i w(b_i) + \Gamma_v$  for appropriate  $n_i$ 's. Now, since  $w$  is exponential,

$$\sum_{i=1}^m n_i w(b_i) + \Gamma_v = \sum_{i=1}^m w(b_i^{n_i}) + \Gamma_v.$$

Letting  $t_i = t(b_i)$  in Lemma 7.2, we see that elements of the form  $\sum_{i=1}^k w(x_i)$  lie in  $\leq \sum_{i=1}^m t_i$  cosets over  $\Gamma_v$  and therefore

$$[M_w : \Gamma_v] \leq \sum_{i=1}^m t_i.$$

Now, using the fact that  $[M_w : \Gamma_v] < \infty$  and the weak cancellation property we show that  $M_w$  is torsion over  $\Gamma_v$ . Let  $\delta \in M_w$ . Since  $[M_w : \Gamma_v] < \infty$ , there exist  $i < j \in \mathbb{N}$  such that  $i\delta + \Gamma_v = j\delta + \Gamma_v$ ; i.e., there exists  $\alpha \in \Gamma_v$  such that  $i\delta = j\delta + \alpha$ . Thus  $i\delta = i\delta + (j-i)\delta + \alpha$  and by the weak cancellation property we have  $(j-i)\delta + \alpha = 0$ , i.e.,  $(j-i)\delta = -\alpha \in \Gamma_v$ . Therefore,  $M_w$  is torsion over  $\Gamma_v$ , and then by Lemma 5.1,  $M_w$  is a group.

Q.E.D.

**Corollary 7.4.** *Let  $w$  be an exponential quasi-valuation with  $M_w$  not necessarily weakly cancellative. The following are equivalent:*

- (1)  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$ .
- (2)  $M_w$  is a torsion group over  $\Gamma_v$ .
- (3)  $M_w$  is weakly cancellative.

*Proof.* (1) $\Rightarrow$ (2). True for any quasi-valuation since if  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$ , then  $M_w$  is torsion over  $\Gamma_v$ . Now, use Lemma 5.1.

(2) $\Rightarrow$ (3). Every group satisfies the weak cancellation property.

(3) $\Rightarrow$ (2). By Lemma 7.3.

(2) $\Rightarrow$ (1). Obvious.

Q.E.D.

We note at this point that since  $M_w$  is a torsion group over  $\Gamma_v$ , one can use the results obtained in sections 5 and 6.

A natural question that arises now is the connection between the quasi-valuation ring  $O_w$  and the integral closures (of  $O_v$  and of  $O_w$ ) inside  $E$ .

**Lemma 7.5.** *If  $w$  is an exponential quasi-valuation, then  $O_w$  contains  $I_E(O_v)$ . In fact,  $O_w = I_E(O_w)$ .*

*Proof.* Let  $x \in I_E(O_w)$  and write  $x^n + \sum_{i=0}^{n-1} \alpha_i x^i = 0$  for  $\alpha_i \in O_w$ . Then

$$nw(x) = w(x^n) \geq \min_{0 \leq i \leq n-1} \{w(\alpha_i x^i)\}$$

$$= w(\alpha_{i_0} x^{i_0}) \geq w(\alpha_{i_0}) + i_0 w(x)$$

for an appropriate  $i_0$ . Therefore

$$(n - i_0)w(x) \geq w(\alpha_{i_0}) \geq 0,$$

i.e.,  $w(x) \geq 0$ .

Q.E.D.

However, if  $w$  is not exponential,  $O_w$  does not necessarily contain  $I_E(O_v)$ . For example, take  $F = \mathbb{Q}$ ,  $v = v_p$ ,  $M = \mathbb{Z}$ ,  $e = i$  and  $\gamma = 1$  in Example 2.9. Note that  $i \notin O_w$  whereas  $x^2 + 1 \in O_v[x]$ .

**Lemma 7.6.** *Let  $w$  and  $w'$  be two exponential quasi-valuations on  $E$  extending  $v$  on  $F$  with value groups  $M_w$  and  $M_{w'}$ . If  $O_{w'} = O_w$  then  $w' = w$ .*

*Proof.* First note that by Lemma 7.3, both  $M_w$  and  $M_{w'}$  are actually torsion groups over  $\Gamma_v$  and thus can be embedded in  $\Gamma_{\text{div}}$ . Assume to the contrary that there exists an  $x \in E$  such that  $w'(x) > w(x)$ ; then, By Lemma 6.2 we get an element  $y \in E$  with  $w'(y) > w(y) = 0$ . Thus,  $y \in I_{w'} \setminus I_w$ . Now, by Corollary 6.6,  $\sqrt{I_w} = J_w$  and  $\sqrt{I_{w'}} = J_{w'}$ ; also, since  $w$  and  $w'$  are exponential quasi-valuations, we have  $\sqrt{I_w} = I_w$  and  $\sqrt{I_{w'}} = I_{w'}$ . However, since  $O_{w'} = O_w$ , we get

$$I_w = \sqrt{I_w} = J_w = J_{w'} = \sqrt{I_{w'}} = I_{w'}$$

which contradicts  $y \in I_{w'} \setminus I_w$ .

Q.E.D

Note that in general  $O_{w'} = O_w$  does not imply  $w' = w$ ; in fact, even for valuations  $v_1$  and  $v_2$  on a field  $F$ ,  $O_{v_1} = O_{v_2}$  only implies the equivalence of  $v_1$  and  $v_2$ . However, in Lemma 7.6 we consider exponential quasi-valuations extending  $v$  where the value groups are embedded in  $\Gamma_{\text{div}}$  (and we get  $O_{w'} = O_w$  implies  $w' = w$ ). In particular, if  $u_1$  and  $u_2$  are valuations on  $E$  extending  $v$  on  $F$  where  $E$  is a finite field extension of  $F$  and  $O_{u_1} = O_{u_2}$ , then  $u_1 = u_2$ .

We recall from section 5 that if  $u_1, \dots, u_n$  are valuations on a field  $E$  which extend a given valuation  $v$  on  $F$ , then  $\min\{u_1, \dots, u_n\}$  is an exponential quasi-valuation on  $E$  extending  $v$ . The corresponding quasi-valuation ring is then the intersection of the valuation rings of the  $u_i$ 's. Such a quasi-valuation ring is integrally closed.

We shall now prove that every exponential quasi-valuation  $w$  extending  $v$  (with  $M_w$  weakly cancellative) is of the form above.

**Theorem 7.7.** *Let  $(F, v)$  be a valued field and let  $w$  be an exponential quasi-valuation extending  $v$  on  $E$ . Then*

$$w = \min\{u_1, \dots, u_k\}$$

for valuations  $u_i$  on  $E$  extending  $v$ .

*Proof.* Let  $w$  be an exponential quasi-valuation. By Lemma 7.5, Remark 5.13 and Remark 5.10,

$$O_w = I_E(O_w) = \bigcap_{i=1}^k O_{u_i}$$

for valuation rings  $O_{u_i}$  where  $u_i$  extends  $v$ . Denote

$$w'(x) = \min\{u_i(x)\}.$$

Then  $w'$  is an exponential quasi-valuation on  $E$  extending  $v$  and  $O_w = O_{w'}$ . Therefore, by Lemma 7.6,  $w = w'$ .

Q.E.D

The following Corollary summarizes the tight connection between exponential quasi-valuations and integrally closed quasi-valuation rings:

**Corollary 7.8.** *Every exponential quasi-valuation  $w$  extending  $v$  induces an integrally closed (quasi-valuation) ring, namely  $O_w$ . Conversely, every integrally closed quasi-valuation ring is of the form  $O_w = I_E(O_w) = \cap_{i=1}^k O_{u_i}$  where each  $u_i$  is a valuation on  $E$  extending  $v$ , and thus  $O_w$  has a (unique) exponential quasi-valuation ( $w = \min\{u_i\}$ ).*

*Proof.* Let

$$A = \{\text{exponential quasi-valuations extending } v\}$$

and

$$B = \{\text{integrally closed quasi-valuation rings}\}.$$

We prove that there is a 1:1 correspondence between  $A$  and  $B$ . We define  $f : A \rightarrow B$  by  $f(w) = O_w$ . Assuming  $w_1 \neq w_2$ , we use Lemma 7.6 to get  $O_{w_1} \neq O_{w_2}$ . Now, if  $O_w \in B$  then, by Remark 5.13 and Remark 5.10,

$$O_w = I_E(O_w) = \cap_{i=1}^k O_{u_i};$$

we take  $w = \min\{u_1, \dots, u_k\}$ .

Q.E.D

**Corollary 7.9.** *There are at most  $\sum_{i=1}^n \binom{n}{i} = 2^n - 1$  exponential quasi-valuations extending  $v$ .*

*Proof.* By Theorem 7.7 every exponential quasi-valuation  $w$  extending  $v$  is of the form  $w = \min\{u_1, \dots, u_t\}$  for valuations  $u_i$  on  $E$  extending  $v$  where the number of these valuations is bounded by  $[E : F] = n$ .

Q.E.D.

We note that although an exponential quasi-valuation induces an integrally closed (quasi-valuation) ring and thus is Bezout (since it is a finite intersection of valuation rings), it is not the case for every quasi-valuation ring. Take for example, again,  $F = \mathbb{Q}$ ,  $v = v_p$ ,  $M = \mathbb{Z}$ ,  $e = i$  and  $n = 1$  in Example 2.9.

The quasi-valuation determines the quasi-valuation ring. Some very important questions that one may consider are: "Is there any sort of converse"? "Is there any sort of canonical quasi-valuation associated to a quasi-valuation ring"? "Is there a ring-theoretic characterization of a quasi-valuation ring"? The answers to these questions, for  $M_w$  a monoid, are affirmative. We shall answer these questions in the next sections.

*From now on we study quasi-valuations extending a valuation on a finite field extension in the general case where the values of the quasi-valuation lie inside a monoid and we do not assume or deduce (as in sections 5,6 and 7) that  $M_w$  (the value monoid) is a group. Namely,  $F$  denotes a field with a valuation  $v$ ,  $E/F$  is a finite field extension with  $n = [E : F]$ , and  $w : E \rightarrow M \cup \{\infty\}$  is a quasi-valuation on  $E$  such that  $w|_F = v$ , where  $M$  is a totally ordered abelian monoid.*

We shall see in section 9 that quasi-valuations can be defined and actually exist in any ring. Moreover, we will show that any subring of  $E$  lying over  $O_v$  has a quasi-valuation extending  $v$ , so the results we get (for quasi-valuation rings with value monoid) apply to all such rings.

## §8 BOUNDING THE QUASI-VALUATION - THE VALUE MONOID CASE

In this section we show that any quasi-valuation extending  $v$  (in a finite field extension  $E/F$ ) is bounded by some valuation on  $E$  extending  $v$ . In order to compare the values of the quasi-valuation with values of a valuation, we construct a total ordering on a suitable amalgamation of  $M_w$  and  $\Gamma_{\text{div}}$ .

*Remark 8.1.* Let  $M$  be a totally ordered abelian monoid, let  $m \in M$ ,  $n \in \mathbb{N}$  and assume  $nm = 0$ ; then  $m = 0$ . Indeed, if  $m > 0$  then  $nm \geq m > 0$ , and thus  $nm > 0$ . If  $m < 0$  the argument is the same.

*Definition 8.2.* A totally ordered monoid  $M$  is called  $\mathbb{N}$ -strictly ordered if  $a < b$  implies  $na < nb$  for every  $n \in \mathbb{N}$ . A totally ordered monoid  $M$  is called *strictly ordered* if  $a < b$  implies  $a + c < b + c$  for every  $c \in M$ .

Note that a totally ordered abelian monoid is strictly ordered iff it is cancellative. Also note that strictly ordered implies  $\mathbb{N}$ -strictly ordered. On the other hand, if  $M$  is a cancellative monoid, then totally ordered,  $\mathbb{N}$ -strictly ordered and strictly ordered are all equivalent. Thus, we need to distinguish among those three types of total ordering only when considering a monoid in general.

From now on until Proposition 8.12 we consider some general properties of a totally ordered abelian monoid  $M$  containing a group  $\Gamma$  (without considering a valuation nor a quasi-valuation).

*Remark 8.3.* If  $m \in M$  is torsion over  $\Gamma$ , then  $m$  is invertible. Indeed if  $nm = \gamma$  then  $-m = (n-1)m - \gamma$ . More generally, if  $nm$  is invertible, then  $m$  is invertible.

*Remark 8.4.* Let  $m, m' \in M$ ,  $n \in \mathbb{N}$  and assume  $nm = nm'$  is invertible; then  $m = m'$ . Indeed by Remark 8.3,  $m$  and  $m'$  are invertible and thus  $n(m - m') = 0$ . We conclude by Remark 8.1 that  $m - m' = 0$ ; i.e.,  $m = m'$ .

*Remark 8.5.* Let  $m, m' \in M$ ,  $n \in \mathbb{N}$  and assume  $nm \leq \gamma \leq nm'$  with  $\gamma$  invertible; then  $m \leq m'$ . (We are done unless  $m \geq m'$ , so  $nm \geq nm'$ , implying  $nm = \gamma = nm'$  is invertible, so  $m = m'$  by Remark 8.4).

*Remark 8.6.* Let  $m, m' \in M$  and assume  $m$  is invertible; then for every  $n \in \mathbb{N}$ ,  $m < m' \Rightarrow nm < nm'$  and  $m > m' \Rightarrow nm > nm'$ . In other words,  $nm \leq nm' \Rightarrow m \leq m'$  and  $nm \geq nm' \Rightarrow m \geq m'$ . Indeed if  $m < m'$  then  $nm \leq nm'$ ; assume to the contrary that  $nm = nm'$ , then, by Remark 8.4, we have  $m = m'$ , a contradiction. The proof of the second assertion is the same.

Recall that  $\Gamma_{\text{div}}$  is the divisible hull of  $\Gamma$ . Also recall that inside  $\Gamma_{\text{div}}$ ,  $(\gamma_1, n_1) = (\gamma_2, n_2)$  iff  $n_2\gamma_1 = n_1\gamma_2$ . We shall now show that one can construct a total ordering on a suitable amalgamation of  $M$  and  $\Gamma_{\text{div}}$  that allows us to compare elements of  $\Gamma_{\text{div}}$  with elements of  $M$ .

Let  $M_{\text{div}}$  denote the divisible hull of  $M$ ; namely,  $M_{\text{div}} = (M \times \mathbb{N}) / \equiv$ , where  $\equiv$  is the equivalence relation defined by  $(m_1, n_1) \equiv (m_2, n_2)$  iff there exists a  $t \in \mathbb{N}$  such that  $tn_2m_1 = tn_1m_2$ . Note that the monoid  $M$  is not necessarily  $\mathbb{N}$ -strictly ordered and thus  $M$  does not necessarily embed in  $M_{\text{div}}$ ; i.e., the function  $\varphi_1 : M \rightarrow M_{\text{div}}$  defined by  $\varphi_1(m) = (m, 1)$  need not be injective. Indeed, we may have  $m_1, m_2 \in M$ ,  $m_1 < m_2$  and  $t \in \mathbb{N}$  such that  $tm_1 = tm_2$  and thus  $(m_1, 1) = (m_2, 1)$  in  $M_{\text{div}}$ . However,  $\Gamma_{\text{div}}$  does embed in  $M_{\text{div}}$  via the function  $\varphi_2 : \Gamma_{\text{div}} \rightarrow M_{\text{div}}$  defined by  $\varphi_2((\gamma, n)) = (\gamma, n)$ , since every element of  $\Gamma$  is invertible. The total ordering in



$M_{\text{div}}$  is defined by:  $(m_1, n_1) \leq (m_2, n_2)$  iff  $tn_2m_1 \leq tn_1m_2$  for some  $t \in \mathbb{N}$ . Note that  $(m_1, n_1) < (m_2, n_2)$  iff for every  $t \in \mathbb{N}$  we have  $tn_2m_1 < tn_1m_2$ . Also note that  $M_{\text{div}}$  is a monoid with the usual addition.

Let  $T$  denote the disjoint union of  $M$  and  $\Gamma_{\text{div}}$ . Let  $\varphi : T \rightarrow M_{\text{div}}$  denote the function defined by:

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{if } x \in M; \\ \varphi_2(x) & \text{if } x \in \Gamma_{\text{div}}. \end{cases}$$

We define an equivalence relation  $\sim$  on  $T$  in the following way: For every  $x, y \in T$ ,

$$x \sim y \text{ iff } \begin{cases} x = y & \text{for } x, y \in M; \\ \varphi(x) = \varphi(y) & \text{otherwise.} \end{cases}$$

It is easy to check that  $\sim$  is indeed an equivalence relation on  $T$ . We shall only check transitivity. Suppose  $x \sim y$  and  $y \sim z$ . The assertion is clear unless  $x, z \in M$  and  $y \in \Gamma_{\text{div}}$ . Write  $x = m, z = m' \in M$ ,  $y = (\gamma, n) \in \Gamma_{\text{div}}$ ; then  $t_1nm = t_1\gamma$  and  $t_2\gamma = t_2nm'$  for some  $t_1, t_2 \in \mathbb{N}$  and thus

$$t_1t_2nm = t_1t_2\gamma = t_1t_2nm'.$$

Hence, by Remark 8.4,  $m = m'$ ; i.e.,  $x \sim z$ .

*Definition 8.7.* Let  $x \in M$ .  $x$  is called *singular* if  $x \sim y$  implies  $x = y$ .

*Remark 8.8.* If  $x, y \in T$  and  $x \sim y$ , then  $x$  is singular iff  $y$  is singular. Indeed, if  $x$  is singular, we get  $x = y$ .

*Remark 8.9.* If  $x \in T$  is not singular then there exists  $(\gamma, n) \in \Gamma_{\text{div}}$  such that  $x \sim (\gamma, n)$ .

*Proof.* If  $x \in \Gamma_{\text{div}}$  the assertion is clear. If  $x \in M$ , then there exists  $y \in T$ ,  $y \neq x$  such that  $x \sim y$ ; by the definition of  $\sim$ ,  $y \in \Gamma_{\text{div}}$ .

Q.E.D.

We denote  $\overline{T} = T / \sim$  and define an order on  $\overline{T}$ : for every  $[x], [y] \in \overline{T}$ ,

$$[x] \leq [y] \text{ iff } \begin{cases} x \leq y & \text{for } x, y \text{ singular;} \\ \varphi(x) \leq \varphi(y) & \text{otherwise.} \end{cases}$$

**Lemma 8.10.**  $\leq$  on  $\overline{T}$  is well defined.

*Proof.* Let  $x, y, a, b \in T$  and assume  $[x] \leq [y]$ ,  $x \sim a$  and  $y \sim b$ ; we prove  $[a] \leq [b]$ . If  $x$  and  $y$  are singular then  $x \leq y$ ,  $x = a$ ,  $y = b$ , and  $a$  and  $b$  are singular; thus  $a \leq b$  and  $[a] \leq [b]$ . If  $x$  and  $y$  are not both singular then  $\varphi(x) \leq \varphi(y)$ , and  $a$  and  $b$  are not both singular. Since  $x \sim a$  and  $y \sim b$  we have  $\varphi(x) = \varphi(a)$  and  $\varphi(y) = \varphi(b)$ ; thus  $\varphi(a) \leq \varphi(b)$  and  $[a] \leq [b]$ .

Q.E.D.

*Remark 8.11.* For  $m \in M$  and  $(\gamma_1, n_1), (\gamma_2, n_2) \in \Gamma_{\text{div}}$  we have the following:

$$\begin{aligned} \varphi(m) \leq \varphi((\gamma_1, n_1)) &\Rightarrow n_1m \leq \gamma_1; \\ \varphi((\gamma_1, n_1)) \leq \varphi(m) &\Rightarrow \gamma_1 \leq n_1m; \\ \varphi((\gamma_1, n_1)) \leq \varphi((\gamma_2, n_2)) &\Rightarrow n_2\gamma_1 \leq n_1\gamma_2. \end{aligned}$$

*Proof.* We shall only prove the first assertion. By the definition of the ordering on  $M_{\text{div}}$ ,  $\varphi(m) \leq \varphi((\gamma_1, n_1)) \Rightarrow tn_1m \leq t\gamma_1$  for some  $t \in \mathbb{N}$ ; by Remark 8.6, we get  $n_1m \leq \gamma_1$ .

Q.E.D.

Note that by Remark 8.11 and by the definition of  $\leq$  on  $\overline{T}$ , one can exclude, for elements of  $\overline{T}$ , the natural number  $t$  in the definition of the ordering on  $M_{\text{div}}$ . For example, assuming  $[m] \leq [(\gamma, n)]$  for  $m \in M$  and  $(\gamma, n) \in \Gamma_{\text{div}}$ . By the definition of  $\leq$  on  $\overline{T}$ ,  $\varphi(m) \leq \varphi((\gamma, n))$ , i.e.,  $(m, 1) \leq (\gamma, n)$  in  $M_{\text{div}}$ . Thus, by the definition of  $\leq$  on  $M_{\text{div}}$ ,  $tnm \leq t\gamma$  for some  $t \in \mathbb{N}$ . However, by Remark 8.11,  $\varphi(m) \leq \varphi((\gamma, n))$  implies  $nm \leq \gamma$ ; so,  $t$  is not needed. We shall use this fact in the proof of the next proposition.

**Proposition 8.12.**  $\leq$  is a total ordering on  $\overline{T}$ .

*Proof.* Reflexivity and antisymmetry follow from the reflexivity and antisymmetry of the total ordering of  $M$  and  $M_{\text{div}}$ . As for transitivity, assume  $[x] \leq [y]$  and  $[y] \leq [z]$ .

If  $x, y, z$  are all singular, the assertion is clear.

If  $x$  and  $y$  are singular and  $z$  is not, write  $x = m_1, y = m_2$  and  $z = (\gamma, n)$ . (Note that by Remark 8.9,  $z \sim (\gamma, n)$  for some  $(\gamma, n) \in \Gamma_{\text{div}}$  and since  $\leq$  is well defined we may write  $z = (\gamma, n)$ ; we shall use this fact throughout the proof). Then  $m_1 \leq m_2$  and  $\varphi(m_2) \leq \varphi((\gamma, n))$ ; i.e.,  $nm_2 \leq \gamma$  and thus

$$nm_1 \leq nm_2 \leq \gamma.$$

Hence  $\varphi(m_1) \leq \varphi((\gamma, n))$ , namely  $\varphi(x) \leq \varphi(z)$ ; thus  $[x] \leq [z]$ .

If  $x$  and  $z$  are singular and  $y$  is not, write  $x = m_1, z = m_2$  and  $y = (\gamma, n)$ . Then  $\varphi(m_1) \leq \varphi((\gamma, n)) \leq \varphi(m_2)$ , which implies that  $nm_1 \leq \gamma$  and  $\gamma \leq nm_2$ . Thus

$$nm_1 \leq \gamma \leq nm_2.$$

Therefore, by Remark 8.5,  $m_1 \leq m_2$ .

If  $y$  and  $z$  are singular and  $x$  is not, write  $y = m_1, z = m_2$  and  $x = (\gamma, n)$ . Then  $\varphi((\gamma, n)) \leq \varphi(m_1)$  and  $m_1 \leq m_2$ ; i.e.,

$$\gamma \leq nm_1 \leq nm_2.$$

Thus  $\varphi((\gamma, n)) \leq \varphi(m_2)$ .

The rest of the cases are not difficult, we shall only prove one of them (the others are proved in the same manner):

If  $x$  is singular and  $y$  and  $z$  are not, write  $x = m, y = (\gamma_1, n_1)$  and  $z = (\gamma_2, n_2)$ . Then  $\varphi(m) \leq \varphi((\gamma_1, n_1)) \leq \varphi((\gamma_2, n_2))$ ; i.e.,  $\varphi(m) \leq \varphi((\gamma_2, n_2))$ .

Q.E.D.

Note that  $\overline{T}$  is not a monoid since no operation has been defined on it.  $\overline{T}$  is merely a set that "preserves" the values of  $M$  and  $\Gamma_{\text{div}}$  and allows us to compare them.

**Lemma 8.13.** *Let  $w$  be a quasi-valuation extending  $v$ ; then there exists a valuation  $u$  of  $E$  extending  $v$  such that  $O_u \supseteq O_w$  and  $I_u \supseteq I_w$ .*

*Proof.* Take  $R = O_w$  in Proposition 5.7; take a maximal ideal  $K$  of  $O_w$  such that  $K \supseteq I_w$ . Now, expand the pair  $(O_w, K)$ .

Note that Lemma 6.1 is a stronger statement than Lemma 8.13; in Lemma 6.1,  $O_u \supseteq O_w$  implies  $I_u \supseteq I_w$  (the existence of such  $O_u$  is shown in Corollary 5.9) whereas in Lemma 8.13 we only have the existence of a valuation ring such that  $O_u \supseteq O_w$  and  $I_u \supseteq I_w$ . The reason is that in section 6 the value monoid of the quasi-valuation is a group.

The proof of the following theorem uses ideas which are close to the ideas of the proofs of Lemma 6.2 and Theorem 6.3. However, one must be very careful since we are not comparing the values inside a group but rather inside  $\overline{T}$ .

**Theorem 8.14.** *Let  $w$  be a quasi-valuation on  $E$  extending a valuation  $v$  on  $F$  and let  $u$  be a valuation on  $E$  extending  $v$  such that  $O_u \supseteq O_w$  and  $I_u \supseteq I_w$ ; then  $u$  dominates  $w$ ; i.e.,  $\forall x \in E, w(x) \leq u(x)$  in  $\overline{T}$  (see above).*

*Proof.* Let  $x \in E$  and assume to the contrary that  $u(x) < w(x)$ ; write  $\sum_{i=0}^n \alpha_i x^i = 0$  for  $\alpha_i \in F$ . Since the sum is zero we must have  $k, l \in \mathbb{N}$  such that  $u(\alpha_k x^k) = u(\alpha_l x^l)$ . Assuming  $k < l$  we have

$$u(x^{l-k} \alpha_l \alpha_k^{-1}) = 0$$

i.e.,  $x^{l-k} \alpha_l \alpha_k^{-1} \notin I_u$ . Now, since  $w(x) > u(x)$ , we have  $w(x^{l-k}) > u(x^{l-k})$ . Indeed,

$$w(x^{l-k}) \geq (l-k)w(x)$$

and

$$(l-k)w(x) > (l-k)u(x) = u(x^{l-k}).$$

Therefore, by the stability of  $\alpha_l \alpha_k^{-1}$  (with respect to  $u$  and  $w$ ), we get,

$$w(x^{l-k} \alpha_l \alpha_k^{-1}) > u(x^{l-k} \alpha_l \alpha_k^{-1}) = 0.$$

Thus  $x^{l-k} \alpha_l \alpha_k^{-1} \in I_w \setminus I_u$ , a contradiction.

Q.E.D.

**Theorem 8.15.** *Let  $w$  be a quasi-valuation of the field  $E$  extending a valuation  $v$ . Then there exists a valuation  $u$  of  $E$  extending  $v$  such that  $O_u \supseteq O_w$ ,  $I_u \supseteq I_w$  and  $u$  dominates  $w$ .*

*Proof.* The existence of a valuation  $u$  with  $O_u \supseteq O_w$  and  $I_u \supseteq I_w$  is by Lemma 8.13. Now, apply Theorem 8.14.

Q.E.D.

In section 10 (Theorem 10.5) we prove a stronger version of Theorem 8.15 for the filter quasi-valuation. We show there that for every valuation  $u$  of  $E$  extending  $v$  such that  $O_u \supseteq O_w$ ,  $u$  dominates the filter quasi-valuation.

## §9 FILTER QUASI-VALUATIONS - EXTENDING VALUATIONS TO QUASI-VALUATIONS

Our goal in this section is to construct a quasi-valuation extending a given valuation. First we obtain a value monoid (we call it the cut monoid), constructed from the value group. We then show that one can extend a valuation to a quasi-valuation with values inside this particular monoid.

### CUTS OF ORDERED SETS

We start this section by reviewing some of the notions of cuts. We shall review the parts needed to construct the cut monoid; for more information about cuts see, for example, [FKK] or [Weh].

In this subsection  $T$  denotes a totally ordered set.

*Definition 9.1.* A subset  $S$  of  $T$  is called initial (resp. final) if for every  $\gamma \in S$  and  $\alpha \in T$ , if  $\alpha \leq \gamma$  (resp.  $\alpha \geq \gamma$ ), then  $\alpha \in S$ .

*Definition 9.2.* A cut  $\mathcal{A} = (\mathcal{A}^L, \mathcal{A}^R)$  of  $T$  is a partition of  $T$  into two subsets  $\mathcal{A}^L$  and  $\mathcal{A}^R$ , such that, for every  $\alpha \in \mathcal{A}^L$  and  $\beta \in \mathcal{A}^R$ ,  $\alpha < \beta$ .

Note that the set of all cuts  $\mathcal{A} = (\mathcal{A}^L, \mathcal{A}^R)$  of the ordered set  $T$  contains the two cuts  $(\emptyset, T)$  and  $(T, \emptyset)$ ; these are commonly denoted by  $-\infty$  and  $\infty$ , respectively. However, we shall not use the symbols  $-\infty$  and  $\infty$  to denote the above cuts since we shall define a "different"  $\infty$ .

Given  $\alpha \in T$ , we denote

$$(-\infty, \alpha] = \{\gamma \in T \mid \gamma \leq \alpha\}$$

and

$$(\alpha, \infty) = \{\gamma \in T \mid \gamma > \alpha\}.$$

One defines similarly the sets  $(-\infty, \alpha)$  and  $[\alpha, \infty)$ .

To define a cut we shall often write  $\mathcal{A}^L = S$ , meaning the  $\mathcal{A}$  is defined as  $(S, T \setminus S)$  when  $S$  is an initial subset of  $T$ .

*Definition 9.3.* The ordering on the set of all cuts of  $T$  is defined by  $\mathcal{A} \leq \mathcal{B}$  iff  $\mathcal{A}^L \subseteq \mathcal{B}^L$  (or equivalently  $\mathcal{A}^R \supseteq \mathcal{B}^R$ ).

Given  $S \subseteq T$ ,  $S^+$  is the smallest cut  $\mathcal{A}$  such that  $S \subseteq \mathcal{A}^L$ . So, for  $\alpha \in T$  we have  $\{\alpha\}^+ = ((-\infty, \alpha], (\alpha, \infty))$ . We denote  $\{\alpha\}^+$  by  $\alpha^+$ .

### THE CUT MONOID

In this subsection,  $\Gamma$  will denote a totally ordered abelian group and  $\mathcal{M}(\Gamma)$  will denote the set of all cuts of  $\Gamma$ .

*Definition 9.4.* Let  $S, S' \subseteq \Gamma$  and  $n \in \mathbb{N}$ , we define

$$S + S' = \{\alpha + \beta \mid \alpha \in S, \beta \in S'\};$$

$$nS = \{s_1 + s_2 + \dots + s_n \mid s_1, s_2, \dots, s_n \in S\}.$$

*Definition 9.5.* For  $\mathcal{A}, \mathcal{B} \in \mathcal{M}(\Gamma)$ , their (left) sum is the cut defined by

$$(\mathcal{A} + \mathcal{B})^L = \mathcal{A}^L + \mathcal{B}^L.$$

One can also define the right sum; however, we shall not use it. Note that under the above definitions, the zero in  $\mathcal{M}(\Gamma)$  is the cut  $0^+ = ((-\infty, 0], (0, \infty))$ .

*Definition 9.6.* For  $\mathcal{A} \in \mathcal{M}(\Gamma)$  and  $n \in \mathbb{N}$ , we define the cut  $n\mathcal{A}$  by

$$(n\mathcal{A})^L = n\mathcal{A}^L.$$

The following lemma is well known (see for example [FKK] or [Weh]).

**Lemma 9.7.**  $(\mathcal{M}(\Gamma), +, \leq)$  is a totally ordered abelian monoid.

*Definition 9.8.* We call  $\mathcal{M}(\Gamma)$  the cut monoid of  $\Gamma$ .

*Remark 9.9.* Note that there is a natural monomorphism of monoids  $\varphi : \Gamma \rightarrow \mathcal{M}(\Gamma)$  defined in the following way: for every  $\alpha \in \Gamma$ ,

$$\varphi(\alpha) = \alpha^+$$

since  $\alpha < \beta \in \Gamma$  implies  $(-\infty, \alpha] \subset (-\infty, \beta]$ . Therefore, when we write  $N \cap \Gamma$  for a subset  $N \subseteq \mathcal{M}(\Gamma)$ , we refer to the intersection of  $N$  with the copy  $\{\alpha^+ \mid \alpha \in \Gamma\}$  of  $\Gamma$ , which could also be viewed as the set  $\{\alpha \in \Gamma \mid \alpha^+ \in N\}$ . To simplify notation, for  $\alpha \in \Gamma$ , when viewing  $\alpha$  inside  $\mathcal{M}(\Gamma)$ , we shall write  $\alpha$  instead of  $\alpha^+$ . For example, for  $\mathcal{B} \in \mathcal{M}(\Gamma)$ ,  $\mathcal{B} + \alpha$  is the cut defined by  $(\mathcal{B} + \alpha)^L = \mathcal{B}^L + (-\infty, \alpha]$ .

*Definition 9.10.* Let  $\alpha \in \Gamma$  and  $\mathcal{B} \in \mathcal{M}(\Gamma)$ . We write  $\mathcal{B} - \alpha$  for the cut  $\mathcal{B} + (-\alpha)$  (viewing  $-\alpha$  as an element of  $\mathcal{M}(\Gamma)$ ).

*Remark 9.11.*  $\mathcal{B} + \alpha$  is actually the cut defined by  $(\mathcal{B} + \alpha)^L = \{\beta + \alpha \mid \beta \in \mathcal{B}^L\}$ .

*Proof.* Obviously  $\{\beta + \alpha \mid \beta \in \mathcal{B}^L\} \subseteq (\mathcal{B} + \alpha)^L$ . For the converse, if  $\gamma \in (\mathcal{B} + \alpha)^L$ , then for some  $\varepsilon \in \mathcal{B}^L$  and  $\delta \in (-\infty, \alpha]$ ,  $\gamma = \varepsilon + \delta \leq \varepsilon + \alpha$ . So,  $\varepsilon \geq \gamma - \alpha$  and thus  $\gamma - \alpha \in \mathcal{B}^L$ , whence,  $\gamma = (\gamma - \alpha) + \alpha \in \{\beta + \alpha \mid \beta \in \mathcal{B}^L\}$ .

**Lemma 9.12.** The cut monoid  $\mathcal{M}(\Gamma)$  is  $\mathbb{N}$ -strictly ordered.

*Proof.* Let  $\mathcal{A}, \mathcal{B} \in \mathcal{M}(\Gamma)$  and assume  $\mathcal{A} < \mathcal{B}$ , i.e.,  $\mathcal{A}^L \subset \mathcal{B}^L$ . Then there exists  $\beta \in \mathcal{B}^L \setminus \mathcal{A}^L$ ; namely  $\beta > \alpha$  for all  $\alpha \in \mathcal{A}^L$ . Thus  $n\beta > \alpha_1 + \alpha_2 + \dots + \alpha_n$  for all  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{A}^L$ ; therefore  $n\beta \in n\mathcal{B}^L \setminus n\mathcal{A}^L$ , i.e.,  $n\mathcal{A} < n\mathcal{B}$ .

Q.E.D.

## CONSTRUCTING THE FILTER QUASI-VALUATION

*Definition 9.13.* Let  $v$  be a valuation on a field  $F$  with value group  $\Gamma_v$ . Let  $O_v$  be the valuation ring of  $v$  and let  $R$  be an algebra over  $O_v$ . For every  $x \in R$ , the  $O_v$ -support of  $x$  in  $R$  is the set

$$S_x^{R/O_v} = \{a \in O_v \mid xR \subseteq aR\}.$$

We suppress  $R/O_v$  when it is understood. Note that  $xR \subseteq aR$  iff  $x \in aR$ .

**Definition 9.14.** Let  $A$  be a collection of sets and let  $B$  be a subset of  $A$ ; we call  $B$  a *filter* of  $A$  if  $B$  satisfies the following:

1. If  $C \in B$  and  $D$  is an element of  $A$  containing  $C$ , then  $D \in B$ ;
2. If  $C, D \in B$  then  $C \cap D \in B$ ;
3.  $\emptyset \notin B$ .

**Remark 9.15.** For every  $x \in R$ , the set  $\{aO_v \mid a \in S_x^{R/O_v}\}$  is a filter of  $\{aO_v \mid a \in O_v\}$ .

For every  $A \subseteq F$  we denote  $v(A) = \{v(a) \mid a \in A\}$ . Note that for  $A \subseteq O_v$ , we have  $v(A) = (v(A))^{\geq 0}$  (viewing  $v(A)$  as a subset of  $\Gamma_v \cup \{\infty\}$ ). In particular,

$$v(S_x) = (v(S_x))^{\geq 0} = \{v(a) \mid a \in S_x\};$$

the reason for this notion is the following observation.

**Lemma 9.16.** *Let  $0 \neq x \in R$ . Then  $v(S_x)$  is an initial subset of  $(\Gamma_v)^{\geq 0}$ .*

*Proof.* First note that  $1 \in S_x$ ; so  $S_x \neq \emptyset$ . Now, let  $\alpha \in v(S_x)$ ; then there exists an  $a \in O_v$  with  $v(a) = \alpha$  such that  $x \in aR$ . Let  $0 \leq \beta \leq \alpha$ ; then there exists a  $b \in O_v$  with  $v(b) = \beta$ . Thus,  $a \in bO_v$  and  $x \in aR \subseteq bR$ . Therefore  $b \in S_x$  and  $\beta \in v(S_x)$ .

Q.E.D.

Note that by Lemma 9.16, we have, for  $0 \neq x \in R$ ,

$$(v(S_x)^+)^L = v(S_x) \cup (\Gamma_v)^{<0}.$$

Also note that if  $A$  and  $B$  are subsets of  $O_v$  such that  $A \subseteq B$  then  $v(A) \subseteq v(B)$ .

Recall that we do not denote the cut  $(\Gamma_v, \emptyset) \in \mathcal{M}(\Gamma_v)$  as  $\infty$ . So, as usual, we adjoin to  $\mathcal{M}(\Gamma_v)$  an element  $\infty$  greater than all elements of  $\mathcal{M}(\Gamma_v)$ ; for every  $\mathcal{A} \in \mathcal{M}(\Gamma_v)$  and  $\alpha \in \Gamma_v$  we define  $\infty + \mathcal{A} = \mathcal{A} + \infty = \infty$  and  $\infty - \alpha = \infty$ .

**Definition 9.17.** Let  $v$  be a valuation on a field  $F$  with value group  $\Gamma_v$ . Let  $O_v$  be the valuation ring of  $v$  and let  $R$  be an algebra over  $O_v$ . Let  $\mathcal{M}(\Gamma_v)$  denote the cut monoid of  $\Gamma_v$ . We say that a quasi-valuation  $w : R \rightarrow \mathcal{M}(\Gamma_v) \cup \{\infty\}$  is *induced by*  $(R, v)$  if  $w$  satisfies the following:

1.  $w(x) = v(S_x)^+$  for every  $0 \neq x \in R$ . I.e.,  $w(x)^L = v(S_x) \cup (\Gamma_v)^{<0}$ ;
2.  $w(0) = \infty$ .

**Remark 9.18.** Notation as in Definition 9.17 and let  $0 \neq r \in R$ . We note that it is possible to have  $v(S_r) = (\Gamma_v)^{\geq 0}$  and thus  $w(r) = (\Gamma_v, \emptyset)$ ; for example, take  $R = F$ . However,  $1 \in S_r$  and thus  $0 \in v(S_r)$ ; therefore  $(v(S_r)^+)^L \supseteq (-\infty, 0]$ ; i.e.,  $w(r) \geq 0$ . Thus, for  $w$  a quasi-valuation induced by  $(R, v)$  and  $r \in R$ , we have  $w(r) \geq 0$  (recall that by definition  $w(0) = \infty$ ). In particular,  $w$  cannot satisfy  $w(r) = (\emptyset, \Gamma_v)$ , i.e.,  $(\emptyset, \Gamma_v) \notin \text{im}(w)$ .

The following theorem holds for arbitrary algebras  $R$  (not necessarily integral domains).

**Theorem 9.19.** *Let  $v$  be a valuation on a field  $F$  with value group  $\Gamma_v$ . Let  $O_v$  be the valuation ring of  $v$  and let  $R$  be an algebra over  $O_v$ . Let  $\mathcal{M}(\Gamma_v)$  denote the cut monoid of  $\Gamma_v$ . Then there exists a quasi-valuation  $w : R \rightarrow \mathcal{M}(\Gamma_v) \cup \{\infty\}$  induced by  $(R, v)$ .*

*Proof.* We define for every  $0 \neq x \in R$  a function  $w : R \rightarrow \mathcal{M}(\Gamma_v) \cup \{\infty\}$  by

$$w(x) = v(S_x)^+;$$

and  $w(0) = \infty$ . Note that by definition  $w$  satisfies conditions 1 and 2 of Definition 9.17. We prove that  $w$  is indeed a quasi-valuation.

First note that if  $x, y \in R$  such that at least one of them is zero then it is easily seen that  $w(xy) \geq w(x) + w(y)$  and  $w(x+y) \geq \min\{w(x), w(y)\}$ ; so, we may assume that  $x, y \in R$  are both non-zero.

Let  $x, y \in R$  and note that  $S_{xy} \supseteq S_x \cdot S_y$  where  $S_x \cdot S_y = \{a \cdot b \mid a \in S_x, b \in S_y\}$ . Indeed, let  $a \in S_x$ ,  $b \in S_y$ ; then  $x = ar$ ,  $y = bs$  for some  $r, s \in R$  and thus  $xy = (ab)(rs)$ . Therefore,

$$v(S_{xy}) \supseteq v(S_x \cdot S_y) = v(S_x) + v(S_y).$$

Hence,

$$\begin{aligned} w(xy) &= v(S_{xy})^+ \geq (v(S_x) + v(S_y))^+ \\ &= v(S_x)^+ + v(S_y)^+ = w(x) + w(y); \end{aligned}$$

i.e.,

$$w(xy) \geq w(x) + w(y).$$

Now, assume  $w(x) \leq w(y)$  i.e.,  $v(S_x) \subseteq v(S_y)$ . Thus  $S_x \subseteq S_y$ . (Indeed, assume to the contrary that there exists an  $a \in S_x \setminus S_y$ ; then for every  $a' \in O_v$  with  $v(a') = v(a) = \alpha$ , one has  $a' \notin S_y$  and thus  $\alpha \notin v(S_y)$ , a contradiction). Consequently, if  $a \in S_x$  we have

$$(x + y)R \subseteq xR + yR \subseteq aR + aR = aR;$$

i.e.,  $a \in S_{x+y}$ . Therefore

$$v(S_{x+y}) \supseteq v(S_x);$$

i.e.,  $v(S_{x+y})^+ \geq v(S_x)^+$ . Thus

$$w(x + y) \geq w(x) = \min\{w(x), w(y)\}.$$

Q.E.D.

*Definition 9.20.* In view of Remark 9.15, the quasi-valuation constructed in Theorem 9.19 is called the *filter quasi-valuation* induced by  $(R, v)$ .

*Example 9.21.* Let  $v$  be a valuation on a field  $F$ . Let  $O_v$  denote the valuation ring and let  $I \triangleleft O_v$ . Then  $O_v/I$  is an algebra over  $O_v$  whose filter quasi-valuation induced by  $(O_v/I, v)$  is:

$$w(x + I) = \begin{cases} v(x) & \text{if } x \notin I \\ \infty & \text{otherwise.} \end{cases}$$

Here  $w : O_v/I \rightarrow \mathcal{M}(\Gamma_v) \cup \{\infty\}$ . Note that in the definition of  $w$  we write  $v(x)$  as an element of  $\mathcal{M}(\Gamma_v)$ , i.e.,  $v(x) = ((-\infty, v(x)], (v(x), \infty))$ . Also note that  $w$  is well defined. We show now that  $w$  is indeed the filter quasi-valuation induced by  $(O_v/I, v)$ . Let  $w_v$  denote the filter quasi-valuation induced by  $(O_v/I, v)$ ; if  $x \in I$

then  $x \equiv 0$  and  $w_v(x + I) = w_v(0) = \infty$ . If  $x \notin I$  then obviously  $w_v(x + I) \geq v(x)$ ; assume that  $w_v(x + I) > v(x)$ . Then there exists  $y \in O_v$  with

$$w_v(x + I) \geq v(y) > v(x)$$

such that  $x + I \in yO_v/I$ ; i.e.,  $x - ya \in I$  for some  $a \in O_v$ . Note that  $v(x) = v(x - ya)$  and thus  $x \in I$ , a contradiction. So,  $w_v = w$ .

Note that if  $I = P$  is a prime ideal of  $O_v$ , then  $w$  above is actually a valuation on  $O_v/P$ .

*Remark 9.22.* A different way to define the filter quasi-valuation described in Example 9.21 is by using the notion of final subsets. Assuming  $v$  is a valuation on a field  $F$  with valuation ring  $O_v$  and value group  $\Gamma_v$ , it is known that there is a one to one correspondence between the set of final subsets of  $\Gamma_v$  and the set of  $O_v$ -submodules of  $F$ . (See [Bo, p. 391-392]; final subsets are called there major subsets). Thus, one can equivalently define

$$w(x + I) = \begin{cases} v(x) & \text{if } v(x) \notin B \\ \infty & \text{otherwise.} \end{cases}$$

where  $B$  denotes the final subset corresponding to  $I$  in Example 9.21.

*Remark 9.23.* Notation as in Theorem 9.19. For every  $c \in O_v$ , we have

$$v(S_{c \cdot 1_R}) \supseteq [0, v(c)],$$

i.e.,  $w(c \cdot 1_R) \geq v(c)$ . Indeed  $c \cdot 1_R \in cR$  and thus  $c \in S_{c \cdot 1_R}$ . Note that one can easily have  $w(c \cdot 1_R) > v(c)$ . For example, take  $0 \neq c \in O_v$  such that  $c \cdot 1_R = 0$  (in particular,  $R$  is not faithful); then  $\infty = w(0) = w(c \cdot 1_R) > v(c)$ . In Example 9.28 we shall see a situation in which  $\infty > w(c) > v(c)$ .

*Remark 9.24.* Let  $R$  be a torsion free algebra over an integral domain  $C$ . Let  $0 \neq c \in C$ ,  $b \in C$  satisfying  $c^{-1}b \in C$ ; let  $x, r \in R$  and assume  $cx = br$ . Since

$$br = c(c^{-1}b)r,$$

we may cancel  $c$  and conclude that  $x = (c^{-1}b)r$ . Note that if  $R$  is not torsion free, this fact is not true, as we shall see in Corollary 9.26.

**Lemma 9.25.** *Notation as in Theorem 9.19, assume in addition that  $R$  is torsion free over  $O_v$ ; then*

$$w(cx) = v(c) + w(x)$$

for every  $c \in O_v$ ,  $x \in R$ .

*Proof.* First note that if  $c = 0$  or  $x = 0$  then  $w(cx) = v(c) + w(x)$  is clear. Now, by Remark 9.23 and by the fact that  $w$  is a quasi-valuation, we have

$$w(cx) = w(c \cdot 1_R \cdot x) \geq w(c \cdot 1_R) + w(x) \geq v(c) + w(x).$$

For the other direction we show that if  $v(b) \in v(S_{cx})$  for  $b \in O_v$ , then

$$v(b) \in [0, v(c)] + v(S_x).$$



Note that if  $v(b) < v(c)$  then clearly  $v(b) \in [0, v(c)] + v(S_x)$ . (Indeed,  $0 \in v(S_x)$  and  $[0, v(c)]$  is an initial subset of  $(\Gamma_v)^{\geq 0}$  and thus  $v(b) \in [0, v(c)]$ ). Thus, we may assume that  $v(b) \geq v(c)$ , i.e.,  $c^{-1}b \in O_v$ . Therefore, by the definition of  $S_{cx}$  and Remark 9.24, we have

$$b \in S_{cx} \Rightarrow cx \in bR \Rightarrow x \in c^{-1}bR.$$

So we have  $c^{-1}b \in S_x$ , and writing  $b = c(c^{-1}b)$ , we conclude that

$$v(b) = v(c) + v(c^{-1}b) \in [0, v(c)] + v(S_x).$$

Q.E.D.

We deduce the following corollary:

**Corollary 9.26.** *Let  $R$  be an algebra over  $O_v$  and let  $w$  denote the filter quasi-valuation induced by  $(R, v)$ ; then  $R$  is torsion free iff*

$$w(cx) = v(c) + w(x)$$

for every  $c \in O_v$  and  $x \in R$ .

*Proof.* ( $\Rightarrow$ ) By Lemma 9.25. ( $\Leftarrow$ ) If  $R$  is not torsion free over  $O_v$ , then there exist  $0 \neq c \in O_v$  and  $0 \neq x \in R$  such that  $cx = 0$ . Hence

$$\infty = w(0) = w(cx) > v(c) + w(x)$$

(since  $v(c), w(x) < \infty$ ).

Q.E.D.

*Remark 9.27.* Note that even in the case where  $R$  is a torsion free algebra over  $O_v$ , one does not necessarily have  $w(c \cdot 1_R) = v(c)$  for  $c \in O_v$ , despite the fact that

$$w(c \cdot 1_R) = v(c) + w(1_R)$$

by Lemma 9.25. The reason is that  $w(1_R)$  is not necessarily 0, as shown in the following example.

*Example 9.28.* Let  $v$  be a valuation on a field  $F$ ,  $O_v$  its valuation ring and let  $O_u \subseteq F$  be any ring strictly containing  $O_v$ ; then  $O_u$  is a valuation ring of  $F$  and  $O_u = S^{-1}O_v$  where  $S = O_v \setminus P$  for some non-maximal prime ideal  $P \triangleleft O_v$ ; see [Bo, section 4] for further discussion. Taking  $R = O_u$  in Theorem 9.19 and noting that  $O_u$  is obviously torsion free over  $O_v$ , we see that for every  $c \in O_v$ , one can write  $c = cs \cdot s^{-1} \in csR$  for every  $s \in S$ . Also, if  $t \notin S$  then  $c \notin ctR$  (since otherwise  $t^{-1} \in R$  which is impossible). Thus

$$v(S_c) = \{\alpha \in (\Gamma_v)^{\geq 0} \mid \alpha \leq v(cs) \text{ for some } s \in S\}.$$

Writing  $H$  for the isolated subgroup corresponding to  $P$ , we deduce that

$$w(c)^L = (-\infty, v(c)] + H^{\geq 0}.$$

In particular,  $w(1)^L = (-\infty, 0] + H^{\geq 0}$ ; thus  $w(1) > 0$  (note that  $P \neq I_v$  and thus  $H \neq \{0\}$ ).

We note that  $(-\infty, 0] + H^{\geq 0} = (-\infty, 0] \cup H^{\geq 0}$ .

Note that in Example 9.28  $w(1) \in \overline{\text{hull}_{\mathcal{M}(\Gamma_v)}(H^{\geq 0})}$  yet  $w(1) > \alpha$  for every  $\alpha \in H^{\geq 0}$  (though  $w$  does not extend  $v$ ); see also Lemma 10.6 for further discussion on such elements.

The next observation is well known.

*Remark 9.29.* Let  $C$  be an integral domain,  $S$  a multiplicative closed subset of  $C$  with  $0 \notin S$ , and  $R$  an algebra over  $C$ . We claim that every  $x \in R \otimes_C CS^{-1}$  is of the form  $r \otimes \frac{1}{\beta}$  for  $r \in R$  and  $\beta \in S$ . Indeed, write  $x = \sum_{i=1}^m (r_i \otimes \frac{\alpha_i}{\beta_i})$  where  $r_i \in R$ ,  $\alpha_i \in C$  and  $\beta_i \in S$ . Let  $\beta = \prod_{i=1}^m \beta_i$  and  $\alpha'_i = \alpha_i \beta \beta_i^{-1} \in C$ . Thus,

$$\sum_{i=1}^m (r_i \otimes \frac{\alpha_i}{\beta_i}) = \sum_{i=1}^m (r_i \otimes \frac{\alpha'_i}{\beta}) = \sum_{i=1}^m (\alpha'_i r_i \otimes \frac{1}{\beta}) = r \otimes \frac{1}{\beta}.$$

Here  $r = \sum_{i=1}^m \alpha'_i r_i$ .

We now consider the tensor product  $R \otimes_{O_v} F$  where  $R$  is a torsion free algebra over  $O_v$ . Our goal is to construct a quasi-valuation on  $R \otimes_{O_v} F$  using the filter quasi-valuation induced by  $(R, v)$  that was constructed in Theorem 9.19.

*Remark 9.30.* Note that if  $R$  is a torsion free algebra over  $O_v$ , then there is an embedding  $R \hookrightarrow R \otimes_{O_v} F$ ; we shall see that in this case the quasi-valuation on  $R \otimes_{O_v} F$  extends the quasi-valuation on  $R$ .

**Lemma 9.31.** *Let  $v, F, \Gamma_v$  and  $O_v$  be as in Theorem 9.19. Let  $R$  be a torsion free algebra over  $O_v$ ,  $S$  a multiplicative closed subset of  $O_v$ ,  $0 \notin S$ , and let  $w : R \rightarrow M \cup \{\infty\}$  be any quasi-valuation where  $M$  is any totally ordered abelian monoid containing  $\Gamma_v$  and  $w(cx) = v(c) + w(x)$  for every  $c \in O_v$ ,  $x \in R$ . Then there exists a quasi-valuation  $W$  on  $R \otimes_{O_v} O_v S^{-1}$ , extending  $w$  on  $R$  (under the identification of  $R$  with  $R \otimes_{O_v} 1$ ), with value monoid  $M \cup \{\infty\}$ .*

*Proof.* In view of Remark 9.29, let  $r \otimes \frac{1}{\beta} \in R \otimes_{O_v} O_v S^{-1}$  and define

$$W(r \otimes \frac{1}{\beta}) = w(r) - v(\beta) \quad (= w(r) + (-v(\beta))).$$

Note that  $W$  is well defined since if  $r \otimes \frac{1}{\beta} = s \otimes \frac{1}{\delta}$  then there exists an  $\alpha \in O_v$  such that  $\alpha(\delta r - \beta s) = 0$  and thus, since  $R$  is torsion free,  $\delta r = \beta s$ . Therefore, by our assumption that  $w(cx) = v(c) + w(x)$  for every  $c \in O_v$  and  $x \in R$ , we have

$$v(\delta) + w(r) = v(\beta) + w(s);$$

i.e.,  $W(r \otimes \frac{1}{\beta}) = W(s \otimes \frac{1}{\delta})$ .

We prove now that  $W$  satisfies the axioms of a quasi-valuation. First note that  $W(0 \otimes 1) = w(0) - v(1) = \infty$ . Next, note that for every two elements  $r \otimes \frac{1}{\beta}, s \otimes \frac{1}{\delta} \in R \otimes_{O_v} O_v S^{-1}$ , assuming that  $v(\beta) \leq v(\delta)$ , we have  $\delta = \alpha\beta$  for some  $\alpha \in O_v$  and thus

$$r \otimes \frac{1}{\beta} = r \otimes \frac{\alpha}{\alpha\beta} = \alpha r \otimes \frac{1}{\delta}.$$

Therefore, we may assume that we have elements  $r \otimes \frac{1}{\delta}, s \otimes \frac{1}{\delta} \in R \otimes_{O_v} O_v S^{-1}$ ; then

$$\begin{aligned} W(r \otimes \frac{1}{\delta} + s \otimes \frac{1}{\delta}) &= W((r + s) \otimes \frac{1}{\delta}) \\ &= w(r + s) - v(\delta) \geq \min\{w(r), w(s)\} - v(\delta) \\ &= \min\{W(r \otimes \frac{1}{\delta}), W(s \otimes \frac{1}{\delta})\} \end{aligned}$$

Now, let  $r \otimes \frac{1}{\beta}, s \otimes \frac{1}{\delta} \in R \otimes_{O_v} O_v S^{-1}$ ; then

$$\begin{aligned} W(r \otimes \frac{1}{\beta} \cdot s \otimes \frac{1}{\delta}) &= W(rs \otimes \frac{1}{\beta\delta}) \\ &= w(rs) - v(\beta\delta) \geq w(r) + w(s) - v(\beta) - v(\delta) \\ &= W(r \otimes \frac{1}{\beta}) + W(s \otimes \frac{1}{\delta}). \end{aligned}$$

Finally note that  $R$  embeds in  $R \otimes_{O_v} O_v S^{-1}$  and

$$W(r \otimes 1) = w(r) - v(1) = w(r).$$

Q.E.D.

**Theorem 9.32.** *Let  $v, F, \Gamma_v, O_v$  and  $\mathcal{M}(\Gamma_v)$  be as in Theorem 9.19. Let  $R$  be a torsion free algebra over  $O_v$  and let  $w$  denote the filter quasi-valuation induced by  $(R, v)$ ; then there exists a quasi-valuation  $W$  on  $R \otimes_{O_v} F$ , extending  $w$  on  $R$ , with value monoid  $\mathcal{M}(\Gamma_v) \cup \{\infty\}$  and  $O_W = R \otimes_{O_v} 1$ .*

*Proof.* Note that by Lemma 9.25,  $w(cx) = v(c) + w(x)$  for every  $c \in O_v, x \in R$ . Thus we can use Lemma 9.31 by taking  $S = O_v \setminus \{0\}$  (and get  $F = O_v S^{-1}$ ). Also note that by Remark 9.29, every  $x \in R \otimes_{O_v} F$  is of the form  $r \otimes \frac{1}{\beta}$  for  $r \in R$  and  $\beta \in O_v$  (by taking  $C = O_v$  and  $S = O_v \setminus \{0\}$ ).

So there exists a quasi-valuation  $W$  on  $R \otimes_{O_v} F$ , extending  $w$  on  $R$ , with value monoid  $\mathcal{M}(\Gamma_v) \cup \{\infty\}$ .  $W$  is given by

$$W(r \otimes \frac{1}{\beta}) = w(r) - v(\beta)$$

for every  $r \otimes \frac{1}{\beta} \in R \otimes_{O_v} F$ .

Finally note that for every element  $r \in R$ , we have, by Remark 9.18, that  $w(r) \geq 0$  and thus  $W(r \otimes 1) = w(r) \geq 0$ . On the other hand, let  $r \otimes \frac{1}{\beta} \in R \otimes_{O_v} F$  with  $W(r \otimes \frac{1}{\beta}) \geq 0$ ; then  $w(r) \geq v(\beta)$  i.e.,  $\beta \in S_r$  and thus one can write  $r = \beta r'$  for some  $r' \in R$ . Hence,

$$r \otimes \frac{1}{\beta} = \beta r' \otimes \frac{1}{\beta} = r' \otimes 1.$$

Consequently,  $O_W = R \otimes 1$ .

Q.E.D.

*Remark 9.33.* Let  $R$  be an algebra over  $O_v$  and let  $0 \neq r \in R$ . By Remark 9.18,  $w(r) \geq 0$ . Thus, for every  $r \otimes \frac{1}{\beta} \in R \otimes_{O_v} F$  where  $r \neq 0$ ,

$$W(r \otimes \frac{1}{\beta}) = w(r) - v(\beta) \geq -v(\beta)$$

i.e.,  $(W(r \otimes \frac{1}{\beta}))^L \neq \emptyset$ . Note that  $(W(0 \otimes \frac{1}{\beta})) = \infty - v(\beta) = \infty$ .

Hence,

$$(\emptyset, \Gamma_v) \notin \text{im}(W).$$

**Theorem 9.34.** *Let  $v, F, \Gamma_v, O_v$  and  $\mathcal{M}(\Gamma_v)$  be as in Theorem 9.19 and let  $A$  be an  $F$ -algebra. Let  $R$  be a subring of  $A$  such that  $R \cap F = O_v$ . Then there exists a quasi-valuation  $W$  on  $RO_v^{-1} = \{rs^{-1} | r \in R, s \in O_v \setminus \{0\}\}$  with value monoid  $\mathcal{M}(\Gamma_v) \cup \{\infty\}$  such that  $R = O_W$  and  $W$  extends  $v$  (on  $F$ ).*

*Proof.* Viewing  $R$  as an algebra over  $O_v$ ,  $R$  has the filter quasi-valuation defined in Theorem 9.19. Note that  $RO_v^{-1} \cong R \otimes_{O_v} F$  and thus, by Theorem 9.32, there exists a quasi-valuation  $W$  on  $RO_v^{-1}$  such that  $R = O_W$ .

We shall now prove that  $W$  extends  $v$ . Note that if  $0 \neq x \in O_v$  then  $x \in S_x$  and thus  $v(S_x) \supseteq [0, v(x)]$  i.e.,  $w(x) \geq v(x)$ . Moreover, for every  $a \in S_x$  one has  $v(a) \leq v(x)$ . (Indeed, if  $x = ar$  for some  $r \in R$  then  $xa^{-1} = r \in O_v$ ; i.e.,  $x \in aO_v$ ). Therefore  $v(S_x) \subseteq [0, v(x)]$  and  $w(x) = v(x)$ . Now, if  $x \in F \setminus O_v$  then  $x = \frac{\alpha}{\beta}$  where  $\alpha, \beta \in O_v$  and by the definition of  $W$ , we have

$$\begin{aligned} W\left(\frac{\alpha}{\beta}\right) &= W\left(\alpha \otimes \frac{1}{\beta}\right) = w(\alpha) - v(\beta) \\ &= v(\alpha) - v(\beta) = v\left(\frac{\alpha}{\beta}\right) \end{aligned}$$

The third equality is since  $\alpha \in O_v$  and as proven before  $w(\alpha) = v(\alpha)$ .

Q.E.D.

We continue our discussion in case  $R$  is an integral domain. We extend  $w$  to  $E$ , the field of fractions of  $R$ . We assume that  $E$  is finite dimensional over  $F$ . Note that  $RO_v^{-1} = \{rs^{-1} | r \in R, s \in O_v \setminus \{0\}\}$  is an integral domain contained in  $E$ ; i.e.,  $RO_v^{-1}$  is an integral domain finite dimensional over  $F$  and is thus a field. Also note that  $RO_v^{-1}$  is a field containing  $R$  and thus contains its field of fractions. Therefore  $RO_v^{-1} = E$ . So every  $x \in E$  can be written as  $rs^{-1}$  where  $r \in R, s \in O_v$ . We have the following important theorem:

**Theorem 9.35.** *In view of Theorem 9.34, the quasi-valuation  $W$ , as defined in Theorem 9.32, is a quasi-valuation on  $E$  extending  $v$ , with  $R = O_W$ . In other words, if  $R \subseteq E$  satisfies  $R \cap F = O_v$  and  $E$  is the field of fractions of  $R$ , then  $R$  and  $v$  induce a quasi-valuation  $W$  on  $E$  such that  $R = O_W$  and  $W$  extends  $v$ .*

Note:  $W$  as described in Theorem 9.35 will also be called the filter quasi-valuation induced by  $(R, v)$ .

Note that the filter quasi-valuation induced by  $(R, v)$  is not necessarily an exponential quasi-valuation. However, we cannot hope for an exponential quasi-valuation in light of the following example.

*Example 9.36.* Let  $v$  denote a  $p$ -adic valuation on  $F = \mathbb{Q}$ ; let  $E = \mathbb{Q}[i]$  and  $R = O_v[pi]$ . Let  $w$  denote a quasi-valuation extending  $v$  with  $O_w = R$ ; then we must have  $w(i) < 0$  (since  $i \notin R$ ) whereas  $w(i^2) = w(-1) = 0$ .

*Example 9.37.* Recall from Lemma 2.8 that if  $w$  is a quasi-valuation on a field  $E$  extending  $v$  on  $F$  and  $[E : F] < \infty$  then for any nonzero  $x \in E$ ,  $w(x) < \alpha$  for some  $\alpha \in \Gamma_v$ . If  $\Gamma_v = \mathbb{Z}$  and  $R \subseteq E$  is a ring such that  $R \cap F = O_v$ , then by Example 9.13 and Remark 9.33,  $M_w$  can be identified as  $\mathbb{Z}$ ; where  $M_w$  is the value monoid of the filter quasi-valuation induced by  $(R, v)$ . Namely, the filter quasi-valuation induced

by  $(R, v)$  is a quasi-valuation extending  $v$  with  $M_w$  a group; therefore  $R = O_w$  has also the properties of a quasi-valuation ring studied in sections 5 and 6.

We summarize the main results we have obtained using the theory of quasi-valuation extending a valuation on a finite field extension.

**Theorem 9.38.** *Let  $v$  be a valuation on a field  $F$  with a valuation ring  $O_v$  and value group  $\Gamma_v$ . Let  $R$  be an integral domain with field of fractions  $E$  finite dimensional over  $F$  and  $R \cap F = O_v$ . Then:*

1. *Every f.g. ideal of  $R$  is generated by  $m \leq [E : F]$  generators.  
(See Corollary 2.4).*
2. *There exists a quasi-valuation  $W$  on  $E$  extending  $v$  such that  $R = O_W$ .  
(See Theorem 9.35)*
3.  *$R$  satisfies LO, INC and GD over  $O_v$ .  
(See Lemma 3.12, Remark 3.5 and Theorem 3.7, and Lemma 4.12 respectively).*
4.  *$K\text{-dim}R = K\text{-dim}O_v$ . (See Corollary 4.13).*
5. *If  $R$  satisfies GU over  $O_v$  then  $R$  satisfies the height formula.  
(See Theorem 4.17).*
6. *If there exists a quasi-valuation  $w$  on  $E$  extending  $v$  with  $R = O_w$  such that  $w(E \setminus \{0\})$  is torsion over  $\Gamma_v$ , then*
  - (a).  *$R$  satisfies GU over  $O_v$ . (See Theorem 5.16).*
  - (b).  *$K\text{-dim}O_v \leq |\text{Spec}(R)| \leq |\text{Spec}(I_E(R))| \leq [E : F]_{\text{sep}} \cdot K\text{-dim}O_v$ .  
(See Theorem 5.21).*
  - (c).  *$R$  has finitely many maximal ideals, the number of which is less or equal to  $[E : F]$ . In fact, for each  $P \in \text{Spec}(O_v)$  there are at most  $[E : F]$  prime ideals  $Q \in \text{Spec}(R)$  lying over  $P$ . (See Theorem 5.19 and Theorem 5.21 and the discussion before Theorem 5.21).*

Note, for example, that in view of Example 9.37, if  $F = \mathbb{Q}$  and  $R$  is as above, then  $R$  satisfies properties 1-6.

## §10 PROPERTIES OF THE FILTER QUASI-VALUATION

In this section we prove some properties of the cut monoid (Lemma 10.1 and Lemma 10.6). These properties are valid in general (for the cut monoid induced by an arbitrary totally ordered abelian group). In addition, we prove some facts regarding filter quasi-valuations extending a valuation on a finite field extension.

**Lemma 10.1.** *Let  $\Gamma$  be a totally ordered abelian group; then the only PIM in the cut monoid  $\mathcal{M}(\Gamma)$  lying over the isolated subgroup  $\{0\}$  is the set  $\{((-\infty, 0], (0, \infty))\}$ ; i.e., the set containing the 0 of  $\mathcal{M}(\Gamma)$ .*

*Proof.* First note that  $\text{hull}_{\mathcal{M}(\Gamma)}(\{0\}) = \{((-\infty, 0], (0, \infty))\}$  lies over  $\{0\}$ . Now, if there exists another PIM  $N \neq \text{hull}_{\mathcal{M}(\Gamma)}(\{0\})$  lying over  $\{0\}$  then take an element  $((-\infty, 0], (0, \infty)) \neq \mathcal{A} \in N$ . Next, take  $0 < \alpha \in \mathcal{A}^L$  and get  $N \cap \Gamma \neq \{0\}$ .

Q.E.D.

Let  $R$  be a ring; we denote by  $J(R)$  its jacobson radical. Note that if  $C \subseteq R$  are commutative rings and  $R$  satisfies GU over  $C$  then every maximal ideal of  $R$  lies over a maximal ideal of  $C$ . In particular, if  $R$  satisfies GU over  $C$  and  $C$  is local then  $J(R) \supseteq J(C)$ .

**Proposition 10.2.** *Let  $v$  be a valuation on a field  $F$  and let  $E/F$  be a finite field extension. Let  $R$  be a subring of  $E$  satisfying  $R \cap F = O_v$  and  $J(R) \supseteq I_v$ . Let  $w$  be the filter quasi-valuation induced by  $(R, v)$  (and thus  $R = O_w$ ). Then*

$$\sqrt{I_w} = J_w (= J(R)).$$

*Proof.* Apply proposition 4.19 by taking  $P = I_v$ ; then use Lemma 10.1, the assumption that  $J(R) \supseteq I_v$  (to deduce  $\sqrt{I_w} \subseteq J_w$ ) and the fact that  $O_w$  satisfies INC over  $O_v$  (to deduce  $\sqrt{I_w} \supseteq J_w$ ).

Q.E.D.

For example, if  $\Gamma_v = \mathbb{Z}$  then, by Example 9.37,  $M_w$  can be identified as  $\mathbb{Z}$  (where  $w$  denotes the filter quasi-valuation induced by  $(R, v)$ ) and thus, by Theorem 5.16,  $R$  satisfies GU over  $O_v$  and therefore  $J(R) \supseteq I_v$ . Hence, by Proposition 10.2,  $\sqrt{I_w} = J(R)$ .

**Lemma 10.3.** *Let  $v$  be a valuation on a field  $F$  and let  $E/F$  be a finite field extension. Let  $R$  be a subring of  $E$  satisfying  $R \cap F = O_v$  and let  $w$  be the filter quasi-valuation induced by  $(R, v)$  (and thus  $R = O_w$ ). Then  $I_w = I_v R$ .*

*Proof.* Let  $x \in I_w$ ; then there exists  $a \in O_v$  with  $v(a) > 0$  such that  $x \in aR$ . Thus,  $x \in aR \subseteq I_v R$ . On the other hand, let  $x \in I_v R$ ; then one can write  $x = ar$  for  $a \in I_v$  and  $r \in R$ . Thus  $w(x) \geq w(a) = v(a) > 0$  and  $x \in I_w$ .

We note now that since  $\mathcal{M}(\Gamma)$  is  $\mathbb{N}$ -strictly ordered, by Lemma 9.12, then one can embed  $\mathcal{M}(\Gamma)$  in its divisible hull  $\mathcal{M}_{\text{div}} = (\mathcal{M}(\Gamma) \times \mathbb{N}) / \sim$  where  $\sim$  is the equivalence relation defined by

$$(m_1, n_1) \sim (m_2, n_2) \text{ iff } n_2 m_1 = n_1 m_2.$$

Note that  $\Gamma_{\text{div}}$  and  $\mathcal{M}(\Gamma)$  embed in this divisible hull.

Now, we aim for a stronger version of Theorem 8.15 when dealing with filter quasi-valuations. This time we may consider the ordering inside the divisible hull of  $\mathcal{M}(\Gamma)$ . We start with the following lemma:

**Lemma 10.4.** *Notation as in Lemma 10.3. Let  $u$  be a valuation on  $E$  extending  $v$  such that  $O_u \supseteq O_w$ ; then  $I_u \supseteq I_w$ .*

*Proof.* By Lemma 10.3,  $I_w = I_v R$ . Thus,

$$I_u = I_u O_u \supseteq I_v O_u \supseteq I_v R = I_w.$$

Q.E.D.

**Theorem 10.5.** *Notation as in Lemma 10.3. Then there exists a valuation  $u$  of  $E$  extending  $v$  on  $F$  such that  $O_u \supseteq O_w$ . Moreover, for every such  $u$ ,  $u$  dominates  $w$ .*

*Proof.* The first part is true for any quasi-valuation, as proved in Lemma 8.13. As for the second part, note that by Lemma 10.4,  $O_u \supseteq O_w$  implies  $I_u \supseteq I_w$  and apply Theorem 8.14.

Q.E.D.

**Lemma 10.6.** *Let  $H$  be an isolated subgroup of a totally ordered abelian group  $\Gamma$ . Then there exist at most two PIMs in  $\mathcal{M}(\Gamma)$  lying over  $H^{\geq 0}$ ; namely,*

$$\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0}) \text{ and } \overline{\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})}.$$

*Proof.* First note that  $H^+ \in \mathcal{M}(\Gamma)$  and  $(H^+)^L = (-\infty, 0] \cup H^{\geq 0}$ . Now, if  $H = \{0\}$  then by Lemma 10.1 there is only one PIM lying over  $H = H^{\geq 0}$ . So, we assume that  $H \neq \{0\}$ . We prove that

$$X = \{\mathcal{A} \in \mathcal{M}(\Gamma) \mid 0 \leq \mathcal{A} < H^+\}$$

is equal to  $\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})$ . We denote

$$Y = \text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0}) \cup \{H^+\} = \{\mathcal{A} \in \mathcal{M}(\Gamma) \mid 0 \leq \mathcal{A} \leq H^+\}.$$

We shall prove that  $Y$  is equal to  $\overline{\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})}$ . Indeed if  $\mathcal{A} \in X$  then there exists  $\alpha \in H^{\geq 0}$  such that  $\alpha \notin \mathcal{A}^L$  and thus

$$(-\infty, 0] \subseteq \mathcal{A}^L \subset (-\infty, \alpha];$$

i.e.,  $\mathcal{A} \in \text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})$ . On the other hand, if  $\mathcal{A} \in \text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})$  then

$$(-\infty, 0] \subseteq \mathcal{A}^L \subseteq (-\infty, \alpha]$$

for  $0 \neq \alpha \in H^{\geq 0}$ ; thus for example,  $\mathcal{A}^L \subset (-\infty, 2\alpha]$  i.e.,  $(\mathcal{A}^L)^{\geq 0} \subset H^{\geq 0}$ . Thus  $\mathcal{A} \in X$ .

Next, if  $\mathcal{A} \in Y$  then obviously  $\mathcal{A} \in \overline{\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})}$ . Now, assuming there exists  $\mathcal{B} \in \overline{\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})} \setminus Y$ , then  $(\mathcal{B}^L)^{\geq 0} \supset H^{\geq 0}$ ; take  $\beta \in (\mathcal{B}^L)^{\geq 0} \setminus H^{\geq 0}$  then  $(-\infty, \beta] \supset H^{\geq 0}$  and thus

$$\overline{\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})} \cap \Gamma \supset H^{\geq 0},$$

a contradiction. Therefore  $\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})$  and  $\overline{\text{hull}_{\mathcal{M}(\Gamma)}(H^{\geq 0})}$  differ only by one element and thus there are no other PIMs lying over  $H^{\geq 0}$ .

Q.E.D.

Here is an example of a group contained in a monoid for which there are more than two PIMs lying over the positive part of an isolated subgroup:

*Example 10.7.* We consider the set  $\mathbb{N} \cup \{0\}$  with the maximum operation, i.e.,  $i + j = \max\{i, j\}$  for all  $i, j \in \mathbb{N} \cup \{0\}$ . Let  $\mathbb{Z} \times (\mathbb{N} \cup \{0\})$  denote the totally ordered abelian monoid with addition defined componentwise and the left to right lexicographic order, i.e.,

$$(z, i) \leq (z', j) \text{ iff } z < z' \text{ or } (z = z' \text{ and } i \leq j).$$

Viewing  $\mathbb{Z}$  inside  $\mathbb{Z} \times (\mathbb{N} \cup \{0\})$  via the natural monomorphism  $z \rightarrow (z, 0)$ , we see that there is an infinite number of PIMs lying over  $\{0\}$ . Namely,

$$\{(0, i)\}_{i \leq j}$$

for every  $j \in \mathbb{N} \cup \{0\}$ .

We present now an example of a quasi-valuation such that inside  $M_w$  there are two PIMs lying over  $\{0\}$ ; in particular, this is not a filter quasi-valuation.

*Example 10.8.* Let  $v$  denote a p-adic valuation on  $\mathbb{Q}$  and let  $M = \{\alpha_0, \alpha_1\}$  be the totally ordered abelian monoid with the maximum operation where  $\alpha_1 > \alpha_0$ . Let  $\mathbb{Z} \times M$  denote the totally ordered abelian monoid defined in a similar way as in Example 10.7 and adjoin a largest element  $\infty$  (as usual). Define  $w$  on  $\mathbb{Q}[\sqrt{p}]$  by

$$w(a + b\sqrt{p}) = \begin{cases} \infty & \text{if } a = b = 0 \\ (v(b), \alpha_1) & \text{if } a = 0, b \neq 0 \\ (v(a), \alpha_0) & \text{if } b = 0, a \neq 0 \\ \min\{(v(a), \alpha_0), (v(b), \alpha_1)\} & \text{if } a \neq 0, b \neq 0. \end{cases}$$

We note that the elements  $(\infty, \alpha_0)$  and  $(\infty, \alpha_1)$  are not defined and thus the case distinction above is needed.

It is not difficult to check that  $w$  is a quasi-valuation on  $\mathbb{Q}[\sqrt{p}]$  extending  $v$  with  $M_w = \mathbb{Z} \times M$  and 2 PIMs over  $\{0\}$ .

Note, for example, that  $w(\sqrt{p}) = (0, \alpha_1) \in \overline{\text{hull}_{M_w}(\{0\})}$  while  $w(\sqrt{p}) > 0$  which is the only element in  $H = \{0\}$  (where obviously  $0 \in \mathbb{Z}$  is identified as  $(0, \alpha_0)$  in  $M_w$ ).

Now, we show that the filter quasi-valuation construction respects localization at prime ideals of  $O_v$ .

*Remark 10.9.* Let  $v$  be a valuation on a field  $F$ , let  $E/F$  be a finite field extension and let  $R \subseteq E$  be a ring such that  $E$  is the field of fractions of  $R$  and  $R \cap F = O_v$ . Let  $w_v$  be the filter quasi-valuation induced by  $(R, v)$ . Thus  $w_v$  extends  $v$  and  $O_{w_v} = R$ . Let  $\mathcal{M}(\Gamma_v)$  denote its cut monoid. Let  $P \triangleleft O_v$  be a prime ideal of  $O_v$ , and  $H$  its corresponding isolated subgroup. Then  $(O_v)_P = S^{-1}O_v$ , where  $S = O_v \setminus P$ .  $S^{-1}O_v$  is a valuation ring containing  $O_v$ ; we denote it by  $O_u$  and its valuation by  $u$ . Recall that for every  $x \in F$ ,  $u(x) = v(x) + H$  and  $\Gamma_u = \Gamma_v/H$ . Note that  $S^{-1}R$  is a subring of  $E$  such that  $S^{-1}R \cap F = O_u$  and thus there exists a filter quasi-valuation, denoted  $w_u$ , induced by  $(S^{-1}R, u)$ ; i.e.,  $w_u$  extends  $u$  and  $O_{w_u} = S^{-1}R$ . We denote its cut monoid by  $M(\Gamma_u) = \mathcal{M}(\Gamma_v/H)$ . Recall that every element  $z \in E$  can be written as  $xy^{-1}$  where  $x \in R$  and  $y \in O_v$ . Also recall that  $\Gamma_v$  embeds in  $\mathcal{M}(\Gamma_v)$  and when we write, for  $y \in O_v$ ,  $v(y)$  as an element of  $\mathcal{M}(\Gamma_v)$  we refer to the cut  $((-\infty, v(y)], (v(y), \infty))$ ; the same notation holds for the valuation  $u$ .

**Lemma 10.10.** *Notation as in Remark 10.9. If  $x \in R$  then*

$$\{v(a) + H \mid a \in S_x^{R/O_v}\} = \{v(b) + H \mid b \in S_x^{S^{-1}R/S^{-1}O_v}\}.$$

*Proof.* ( $\subseteq$ ) If  $a \in S_x^{R/O_v}$  then  $x \in aR$  for  $a \in O_v$  and obviously  $x \in aS^{-1}R$  and  $a \in S^{-1}O_v$ , thus  $a \in S_x^{S^{-1}R/S^{-1}O_v}$ . ( $\supseteq$ ) If  $b \in S_x^{S^{-1}R/S^{-1}O_v}$  then  $x \in bS^{-1}R$  for  $b \in S^{-1}O_v$ . We have two possibilities:

Case I.  $v(b) < 0$ , then  $v(b) \in H$  (since otherwise  $b \notin S^{-1}O_v$ ) and we are done.



Case II.  $v(b) \geq 0$ , assume to the contrary that  $v(b) > v(a) + h$  for every  $a \in S_x^{R/O_v}$  and  $h \in H$ ; we have  $x = bs^{-1}r$  for  $s \in S$ ,  $r \in R$  and thus  $v(bs^{-1}) > 0$  (since  $v(b) > v(a) + h \geq h$  for every  $h \in H$ ) i.e.,  $bs^{-1} \in O_v$ .

Therefore, writing  $a = bs^{-1} \in S_x^{R/O_v}$ , we have a contradiction (since  $v(b) = v(a) + v(s)$ ).

Q.E.D.

**Theorem 10.11.** *Let  $w : E \rightarrow \mathcal{M}(\Gamma_v/H)$  be the function defined by*

$$w(z) = w(xy^{-1}) = \{v(a) + H \mid a \in S_x^{R/O_v}\}^+ + (-v(y) + H),$$

$\forall z \in E$  where  $z = xy^{-1}$  for  $x \in R$  and  $0 \neq y \in O_v$ . Then  $w$  is the filter quasi-valuation induced by  $(S^{-1}R, u)$ ; i.e.,  $w = w_u$ .

*Proof.* Let  $z \in E$  and write  $z = xy^{-1}$  for  $x \in R$  and  $0 \neq y \in O_v$ . Note that  $y \in O_v \subseteq S^{-1}O_v$  is stable with respect to  $w_u$  and  $x \in R \subseteq S^{-1}R$ ; thus

$$w_u(z) = w_u(xy^{-1}) = w_u(x) - w_u(y) = w_u(x) - u(y).$$

Now,

$$\begin{aligned} w_u(x) - u(y) &= u(S_x^{S^{-1}R/S^{-1}O_v})^+ - u(y) \\ &= \{u(b) \mid b \in S_x^{S^{-1}R/S^{-1}O_v}\}^+ - u(y). \end{aligned}$$

Note that  $b \in O_u$  and  $u(b) = v(b) + H$ . Moreover, by Lemma 10.10,

$$\{v(b) + H \mid b \in S_x^{S^{-1}R/S^{-1}O_v}\} = \{v(a) + H \mid a \in S_x^{R/O_v}\}.$$

Also,  $y \in O_v$  and  $u(y) = v(y) + H$ . Consequently,  $w = w_u$ .

Q.E.D.

We shall now present the minimality of the filter quasi-valuation with respect to a natural partial order.

For every ring  $R \subseteq E$  satisfying  $R \cap F = O_v$ , we denote by

$$\mathcal{W}_R = \{w \mid w \text{ is a quasi-valuation on } E \text{ extending } v \text{ with } O_w = R\}.$$

Note that  $\mathcal{W}_R$  is not empty by Theorem 9.35.

We define an equivalence relation on  $\mathcal{W}_R$  in the following way:  $w_1 \sim w_2$  iff for every  $x, y \in E$ ,

$$w_1(x) < w_1(y) \iff w_2(x) < w_2(y).$$

We define a partial order on  $\mathcal{W}_R/\sim$  in the following way:  $[w_1] \leq [w_2]$  iff for every  $x, y \in E$ ,  $w_1(x) < w_1(y)$  implies  $w_2(x) < w_2(y)$  (we say that  $w_1$  is coarser than  $w_2$  and that  $w_2$  is finer than  $w_1$ ). Note that  $\leq$  is well defined and is indeed a partial order on the set of equivalent quasi-valuations corresponding to  $R$ .

We shall prove now that the equivalence class of the filter quasi-valuation is the minimal one with respect to the partial order defined above.

**Proposition 10.12.** *The (equivalence class of the) filter quasi-valuation is the coarsest of all (equivalence classes of) quasi-valuations in  $\mathcal{W}_R/\sim$ .*

*Proof.* We first prove that if  $x, y \in R$  and  $w \in \mathcal{W}_R$  then  $w_v(x) < w_v(y)$  implies  $w(x) < w(y)$  (where  $w_v$  is the filter quasi-valuation induced by  $(R, v)$  and thus  $O_{w_v} = R$ ). Let  $x, y \in R$ ,  $w \in \mathcal{W}_R$  and assume  $0 \leq w_v(x) < w_v(y)$ ; then there exists an  $\alpha \in \Gamma_v$  such that

$$w_v(x) < \alpha \leq w_v(y).$$

Let  $a \in F$  with  $v(a) = \alpha$ ; then  $w_v(ya^{-1}) \geq 0$  and thus  $ya^{-1} \in R$  and  $w(ya^{-1}) \geq 0$ , i.e.,  $w(y) \geq v(a)$ . Note that  $w(x) < v(a)$  since otherwise  $xa^{-1} \in R$  and  $w_v(x) \geq v(a)$ , a contradiction.

Now, in the general case, let  $x, y \in E$ ,  $w \in \mathcal{W}_R$  and assume that  $w_v(x) < w_v(y)$ . Write

$$w_v(x) = \mathcal{A} - \alpha, \quad w_v(y) = \mathcal{A}' - \alpha'$$

for  $\mathcal{A}, \mathcal{A}' \in \mathcal{M}(\Gamma_v)$ ,  $\mathcal{A}, \mathcal{A}' \geq 0$  and  $\alpha, \alpha' \in (\Gamma_v)^{\geq 0}$ ; then  $0 \leq \mathcal{A} < \mathcal{A}' - \alpha' + \alpha$ . Let  $a \in F$  with  $v(a) = \alpha$  and define  $x' = xa$ ,  $y' = ya$ ; thus

$$0 \leq w_v(x') = \mathcal{A} < w_v(y') = \mathcal{A}' - \alpha' + \alpha$$

and, by the proof of the first part,  $w(x') < w(y')$ . Therefore

$$w(x) = w(x') - v(a) < w(y') - v(a) = w(y).$$

Q.E.D.

I would like to thank S. Margolis, P. J. Morandi, J.-P. Tignol, U. Vishne, and A.R. Wadsworth for their helpful comments and queries. I would also like to thank the referee for his helpful comments and suggestions.

Special thanks are, of course, to my advisor, L. H. Rowen, who spent hours, with and without me, going through the ideas presented in this paper. Rowen suggested many crucial corrections, improvements and questions. I owe a very special debt to him.

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